Selected Recitation Notes for Advanced Calculus

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This will be a quick review on the materials you might have seen before.

Let us start with a sentence: $\mathbb{R}$ *forms a complete ordered field which contains* $\mathbb{Q}$ *as a dense subset.* We shall describe properties of $\mathbb{R}$ with axioms which fall into three main categories: algebraic properties, ordering properties and completeness. I am going to omit the first two here please check the notes carefully. And the completeness will be covered in lectures with associated properties and exercises covered in the further recitations.

Consider the set of natural number $\mathbb{N} = \{1, 2, \cdots\}$, it satisfies the property that $1 \in \mathbb{N}$ and if $n \in \mathbb{N}$ then $n+1 \in \mathbb{N}$. We get all natural numbers by adding one to $1 \in \mathbb{N}$. That allows us to formulate induction. Together with 0, $\mathbb{N} \cup \{0\}$ with addition forms a monoid, a semigroup with additive identity 0. By adding more elements (additive inverses), we arrive at $\mathbb{Z}$, the set of integers, and $(\mathbb{Z}, +, \cdot)$ is a commutative ring with identity. By including multiplicative inverses, we get $\mathbb{Q}$. The rational numbers, give us a nice example regarding to the notion of equivalence relation.

Let us first recall some of the concepts.

Definition 1.1. Let X and Y be two sets. A relation R is a subset of the cartesian product $X \times Y$ consisting of ordered pairs (x, y) where $x \in X$ and $y \in Y$. We often write xRy instead of $(x, y) \in R$ if x and y are related.

Both equivalence relation and function are special types of relation.

Definition 1.2. A relation $\equiv$ on a set X is called an equivalence relation if it is reflexive, symmetric and transitive.

Once can check that the set $\mathbb{Q}$ of rational numbers is a collection of equivalence classes in $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ under the equivalence relation $(a, b) \sim (a', b')$ if $ab' = ba'$.

Definition 1.3. A function $f : X \to Y$ is a subset $f \subseteq X \times Y$ such that for any $x \in X$ there is exactly one element $y \in Y$ such that $(x, y) \in f$. We often write $y = f(x)$.

We call the set X on which the function f is defined domain, and the set Y codomain. Here are some other notations:

- The range of image of f:
$$f(X) = \{y \in Y \mid \text{there is } x \in X \text{ such that } f(x) = y\} \subseteq Y.$$
- The (direct) image of a subset $A \subseteq X$:
$$f(A) = \{y \in Y \mid \text{there is } a \in A \text{ such that } f(a) = y\} \subseteq f(X).$$
- The inverse image of a subset $B \subseteq Y$:
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X.$$

  Note that this notation is defined even if the inverse of f does not exist.

- The graph of f is the subset of $X \times Y$ defined by

$$\{(x, f(x)) \mid x \in X\}.$$

Here are some simple examples:

Example 1.4.
- Identity function on a set X; Let $A \subset X$, the indicator function of A, $\chi_A : X \to \{0, 1\}$ is given by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A. \end{cases}$$

- Is $g : \mathbb{Q} \to \mathbb{Q}$ given by $g(a/b) = ab$ a function? Why?

Now you may want to think about that is the interaction between f, f^{-1} and set operations, for instance, taking unions and intersections. Here is an theorem encoding the relations:

Theorem 1.5. *Given a function $f : X \to Y$, then for any $A, B \subseteq X$ and $C, D \subseteq Y$, we have*
(1) $f(A \cup B) = f(A) \cup f(B)$.
(2) $f(A \cap B) \subseteq f(A) \cap f(B)$.
(3) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
(4) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Proof. I will only prove (1) here. In order to prove two sets are the same, we only need to show two inclusions hold. For any $y \in f(A \cup B)$, there exists some $x \in A \cup B$ such that $f(x) = y$. Since $x \in A \cup B$, either $x \in A$ or $x \in B$. If $x \in A$, then $y \in f(A)$; if $x \in B$, then $y \in f(B)$. Thus $y \in f(A) \cup f(B)$. For the other direction, for any $y \in f(A) \cup f(B)$, we have $y \in f(A)$ or $y \in f(B)$. Again, if $y \in f(A)$, then there exists $a \in A$ such that $f(a) = y$; if $y \in f(B)$, then there exists $a \in B$ such that $f(a) = y$. So there exists $a \in A \cup B$ such that $f(a) = y$. $\square$

Remark 1.6. One should be careful that in general, we don't have the equality on (2). Here is an example. Let $A = \{-1, 0, 1\}$, $B = \mathbb{Z}$ and define the function $f : A \to B$ to be $f(n) = n^2$. Take

$$A = \{-1, 0\}, \quad B = \{0, 1\}.$$

Then $A \cap B = \{0\}$ and thus $f(A \cap B) = \{0\}$. But $f(A) \cap f(B) = \{0, 1\} \neq \{0\}$. What if the function f is injective?

Now recall a function $f : X \to Y$ being injective, surjective and bijective.

Definition 1.7. A function $f : X \to Y$ is called
- surjective or onto, if for any $y \in Y$, there is $x \in X$ such that $f(x) = y$.

- injective or one-to-one, if for any distinct $x_1, x_2 \in X$, we have $f(x_1) \neq f(x_2)$. Or equivalently, if for any $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- bijective if f is onto and one-to-one, *i.e.* for any $y \in Y$ there is a unique $x \in X$ such that $f(x) = y$.

One good thing about bijections is that they have inverses. This will turn the following set

$$S_X = \{f : X \to X \text{ bijection}\}$$

with composition as the operation, into a group.

Recall that the composition of two functions $f : X \to Y$ and $g : Y \to Z$ is the function $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$. The order here is important, and be careful about the domains and ranges. Given a function $f : X \to Y$, if there exists a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$, then we call such g the inverse function of f. We have the following characterization of onto, one-to-one and bijective functions:

Theorem 1.8. *Given a function $f : X \to Y$.*

- *f is onto if and only if a right inverse exists,* i.e. *there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.*
- *f is injective if and only if a left inverse exists,* i.e. *there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.*
- *f is bijective if an only if its inverse exists.*

Proof. To prove this theorem, suppose f is onto, then for any $y \in Y$ there exists $x \in X$ such that $f(x) = y$. Define $g(y) = x$. Then $f \circ g(y) = f(x) = y$ is the identity function on Y. Note that right inverse may not be unique! It is unique only if f is a bijection. Conversely if there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$. For any $y \in Y$, let $x = g(y)$. Then $y = f(g(y)) = f(x)$ proves the function f is onto. The rest is left as an exercise. $\square$

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One of the applications for using bijection is to show two sets contains same "number" of elements. Indeed, this is how we define a set E to be countable if we could build up a one-to-one correspondence between E and the set of natural number $\mathbb{N}$.

Definition 2.1. A set E is called countably infinite if there is a bijection $f : E \to \mathbb{N}$. A set if called uncountable if it is not countably infinite or finite.

We call a nonempty set A finite if there exists some $n \in \mathbb{N}$ and a bijection $f : A \to [n]$. I will use the terminology "countable" to represent countably infinite. But some of the textbooks are using "countable" to represent either finite or countably infinite.

The following are some properties of countable sets.

Proposition 2.2. *Given two nonempty sets A, B.*
(1) *If A is countable and $B \subseteq A$, then either B is finite or countable.*
(2) *If A is countable and B is either countable or finite, then $A \cup B$ is countable.*
(3) *If $A_1, A_2, \cdots$ is a sequence of countable sets, then $\cup_{j=1}^{\infty} A_j$ is countable.*
(4) *If $B \subseteq A$ and B is uncountable, then A is uncountable.*

Proof. I will only prove (1) in detail. To see this, we can enumerate a countable set $A = \{a_1, a_2, \cdots\}$ (what is the bijection to $\mathbb{N}$?). Let $B \subseteq A$ be a nonempty subset. Then write

$$B = \{n \in \mathbb{N} \mid a_n \in B\} \subset \mathbb{N}$$

to keep track on all the indices of $a_n \in B$. Then by the well-ordering property, B has a least element, denoted by n_1. Then consider

$$\{n \in \mathbb{N} \mid a_n \in B, n > n_1\}.$$

If it is empty, then B is finite; if not, repeat this process by applying well-ordering property. Thus we can define a function $f : \mathbb{N} \to B$ by $k \mapsto a_{n_k}$ which is a bijection.

To see (2), we only need to prove the disjoint version and interpret $A \cup B = A \cup (B \setminus A)$ where $A \cap (B \setminus A) = \emptyset$. And the case when $A \cap B = \emptyset$ can be proved with the help of the countability of sets of even/odd numbers. $\square$

Example 2.3.
- $\mathbb{N}$ is itself countable by the identity function $\mathrm{id} : \mathbb{N} \to \mathbb{N}$.
- $\mathbb{Z}$ is countable since

$$f(n) = \begin{cases} 2n & \text{if } k \in \mathbb{Z}_{>0}; \\ -2n+1 & \text{if } k \in \mathbb{Z}_{\leq 0} \end{cases}$$

  is a bijection.
- The cartesian product $\mathbb{N} \times \mathbb{N}$ is countable.
  One can use either the diagonal argument, or construct a function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by sending (m, n) to $2^m 3^n$. Since the prime factorization is unique, so the function is clearly injective. But this function is definitely not onto since there is no preimage for, say 7 (natural numbers with primes other than $2, 3$ as divisors). Since $\mathbb{N} \times \mathbb{N}$ is not finite, it has to be countable.
- $\mathbb{Q}$ is countable. There are multiple ways to see this. We can consider each rational number with of form p/q with p, q coprime and $q > 0$ and consider the function $\mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ by $p/q \mapsto (p, q)$. Or one can also define $A_n = \{m/n \mid m \in \mathbb{Z}\}$ for each $n \in \mathbb{N}$ with map $\mathbb{Z} \to A_n$ given by $m \mapsto m/n$ a bijection, thus each A_n is countable. Thus $\cup_{n=1}^{\infty} A_n = \mathbb{Q}$ is countable.
- $\mathbb{R}$ is uncountable. This can be proved by showing that the interval $(0, 1)$ is uncountable. (If $(0, 1)$ is countable, then there exists a bijection $f : \mathbb{N} \to (0, 1)$. Consider the decimal representations and find an element in $(0, 1)$ with no preimage.)

Now let us move on to have a quick review of supremum and infimum of a set of real numbers. Given a set $E \subseteq \mathbb{R}$, if $M \in \mathbb{R}$ (may not be an element in E) is an upper bound of E such that $M \leq M'$ for any upper bound M' of E, then we call M a least upper bound or supremum of E, denoted by $\sup E$. Likewise, if $m \in \mathbb{R}$ is a lower bound of E such that $m' \leq m$ for any lower bound m' of E, then we call m a greatest lower bound or infimum of E. And we have seen in lecture that if sup exists, it is unique, and properties of inf follows from that of sup. And moreover, if both $\sup E$ and $\inf E$ exist, then we have the inequality $\inf E \leq x \leq \sup E$ for any $x \in E$.

Given a nonempty subset $E \subset \mathbb{R}$.

The completeness axiom guarantees that every nonempty subset of $\mathbb{R}$ which is bounded from above has a supremum. If E is bounded from below, consider

$$-E = \{-x \mid x \in E\},$$

then it is equivalent to that $-E$ is bounded from above, and thus $\sup(-E)$ exists. So with the reflection property $\inf E = -\sup(-E)$ (prove it) we get every bounded from below nonempty subset of $\mathbb{R}$ has a infimum. This is a direct consequence of completeness axiom.

Other results from completeness axiom include Archimedean property, and density of $\mathbb{Q}$, etc. Furthermore, sometimes it is useful by using the approximation of supremum. Namely, suppose $E \subset \mathbb{R}$ is bounded from above and $S \in \mathbb{R}$, then $S = \sup E$ if and only if for any $\epsilon > 0$, there exists some $a \in E$ such that $S - \epsilon < a \leq S$. Write down the infimum version of approximation theorem.

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The other way I like to interpret the infimum is the following. Use the same assumption by assuming $E \subset \mathbb{R}$ is nonempty and bounded from below. Consider the set of all the lower bounds of E:

$$L := \{l \in \mathbb{R} \mid l \leq x \text{ for all } x \in E\}.$$

Then set $L \subseteq \mathbb{R}$ is nonempty is bounded above. By the completeness axiom, $\alpha = \sup L$ exists. Claim that the supremum $\alpha = \inf E$. For any $x \in E$, x is an upper bound of L, thus $\alpha \leq x$ meaning that α is a lower bound of E. For any lower bound l of E, since $l \in L$ we must have $l \leq \alpha$. Thus α is the greatest lower bound of E. Therefore $\alpha = \sup L = \inf E$.

Let's look at some examples.

Example 3.1.

- $E = \left\{x \in \mathbb{R} \mid x = 1 + \frac{(-1)^n}{n}, n \in \mathbb{N}\right\}$.

  Depending on whether n is even or odd, we can write

$$x = \begin{cases} 1 + \frac{1}{n} & \text{if } n \text{ is even;} \\ 1 - \frac{1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

Note that for any $n \in \mathbb{N}$, $1 < 1 + 1/n \leq 2$ while $0 \leq 1 - 1/n < 1$. Thus the set E is both bounded from above and below. Actually, the minimum attains at $n = 1$ (odd) and thus $\inf E = \min E = 0$. Since $\{1 + 1/n\}$ is decreasing, then the maximum attains at $n = 2$, with $\sup E = \max E = 1 + 1/2 = 3/2$.

- $E = [0, \sqrt{2}] \cap \mathbb{Q}$.
  First notice that 0 and $\sqrt{2}$ can be viewed as lower bound and upper bound of set E, respectively. Moreover, $\inf E = \min E = 0$. Now claim that $\sqrt{2}$ is the supremum of E. If there is an upper bound satisfying $x \leq \sqrt{2}$, then apply the density of $\mathbb{Q}$ there must be a rational number $q \in \mathbb{Q}$ such that $x \leq q \leq \sqrt{2}$ violating that x is an upper bound. Thus $\sqrt{2}$ must be the least upper bound.
- $E = \left\{n + \frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{10\}$.
  First of all E is not bounded from above since if N is an upper bound of E, then $N < ([N] + 1) + \frac{1}{[N]+1} \in E$ is a contradiction. E is clearly bounded from below. Since for any $n \in \mathbb{N}$, we have $n + \frac{1}{n} \geq 2$ (simple AM-GM) with inequality holds when $n = 1$. Thus $\inf E = \min E = 2$.
- $E = \left\{n + \frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{1\}$. As the example above, E is not bounded from above. And $\min\{n + 1/n \mid n \in \mathbb{N}\} = 2$. Thus $\inf E = \min E = \min\{2, 1\} = 1$.

Enlightened by the last example above, we can formulate in general:

Proposition 3.2. *Given $A, B \subset \mathbb{R}$ bounded. Then $A \cup B$ is bounded and* $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

Here are some other properties of sup/inf.

Proposition 3.3 (Monotonicity)**.** *Suppose $A \subseteq B$ are two nonempty subsets of $\mathbb{R}$.*
(1) If B is bounded from above, then $\sup A \leq \sup B$.
(2) If B is bounded from below, then $\inf A \geq \inf B$.

Proof. I will only prove (1). If B is bounded from above, then any upper bound of B is also an upper bound of A thus A is also bounded from above. Both $\sup A$ and $\sup B$ exist by completeness axiom. Moreover $\sup B$ is an upper bound of A thus $\sup A \leq \sup B$. □

Let $A, B \subset \mathbb{R}$. Define

$$A + B := \{a + b \mid a \in A, b \in B\},$$
$$A - B := \{a - b \mid a \in A, b \in B\}.$$

Proposition 3.4. *If A, B are nonempty subsets of $\mathbb{R}$.*
(1) $\sup(A + B) = \sup A + \sup B$.
(2) $\sup(A - B) = \sup A - \inf B$.
Write down the version for infimums yourself.

Proof. First of all $A + B$ is bounded from above if and only if A and B are both bounded from above. Thus $\sup(A + B)$ exists if and only if $\sup A$ and $\sup B$ both exist. In this case, for any $a \in A$ and $b \in B$, we have

$$a + b \leq \sup A + \sup B.$$

It implies $\sup(A + B) \leq \sup A + \sup B$. For the other direction, apply the approximation theorem. For any $\epsilon > 0$. There are $a \in A$ and $b \in B$ such that

$$\sup A - \frac{\epsilon}{2} < a, \quad \sup B - \frac{\epsilon}{2} < b.$$

Thus $\sup A + \sup B - \epsilon < a + b$ for any $\epsilon > 0$. Therefore we get $\sup A + \sup B \leq \sup(A + B)$.

(2) can be proved using infection property the result from (1). Namely $\sup(A - B) = \sup A + \sup(-B) = \sup A - \inf B$. ☐

4 September 1

We can also define supremum and infimum of a real-valued function, *i.e.* a function with range a subset in $\mathbb{R}$, by taking the sup/inf of its range. Here is the definition.

Definition 4.1. Suppose $f : A \to \mathbb{R}$ is a one-variable real-valued function with bounded range. Then

$$\sup_A(f) = \sup\{f(x) \mid x \in A\},$$
$$\inf_A(f) = \inf\{f(x) \mid x \in A\}.$$

Straightforward properties are

Proposition 4.2. *Suppose that $f, g : A \to R$ and $f \leq g$ pointwisely,* i.e. $f(a) \leq g(a)$ *for any $a \in A$.*

- *If g is bounded from above, then* $\sup_A(f) \leq \sup_A(g)$*;*
- *If f is bounded from below, then* $\inf_A(f) \leq \inf_A(g)$*.*

Proof. If $f \leq g$ pointwisely and g is bounded from above on A, then for any $a \in A$,

$$f(a) \leq g(a) \leq \sup_A(g).$$

Thus $\sup_A(g)$ is an upper bound of $f(A)$ meaning $\sup_A(f) \leq \sup_A(g)$. ☐

Remark 4.3.

- the condition $\sup_A(f) \leq \sup_A(g)$ cannot give any information on the ordering of values $f(a)$ and $g(a)$ for $a \in A$. Also, even if we have the strict inequality $f < g$ pointwisely, it doesn't mean we get $\sup_A(f) < \sup_A(g)$ strictly. For example let $A = (-1, 1)$ and $g \equiv 1$ while $f(x) = |x|$.

- In general, if we only have $f \le g$ pointwisely (weak), we don't have $\sup(f) \le \inf(g)$. For example let $f, g : [0,1] \to \mathbb{R}$ be given by $f(x) = 3x$ and $g(x) = 3x+1$. Clearly we have $f(x) \le g(x)$ for any $x \in [0,1]$. But $\sup(f) = 3, \inf(f) = 1$.
  But if we assume $f(x) \le g(y)$ for any $x, y \in A$, then we get $\sup(f) \le \inf(g)$. It is much clearer by drawing a graph.
  Let me sketch the proof here. By way of contradiction if we have $\sup(f) > \inf(g)$. Then for any $\epsilon > 0$, there are $x, y \in A$ such that $\sup(f) - \epsilon < f(x)$ and $g(y) < \inf(g) + \epsilon$, *i.e.*
  $$g(y) - \epsilon < \inf(g) < \sup(f) < f(x) + \epsilon$$
  meaning
  $$g(y) - f(x) - 2\epsilon < \inf(g) - \sup(f).$$
  By taking $\epsilon = (\sup(f) - \inf(g))/2$ we have $g(y) - f(x) < 0$ for some $x, y \in A$ which is a contradiction.

Given two functions $f, g : A \to \mathbb{R}$, Define

$$f + g := \{f(x) + g(x) \mid x \in A\}.$$

What is the order relation between $\sup_A(f+g)$ and $\sup_A f + \sup_A g$? Note that if we view $F = \{f(x) \mid x \in A\}$ and $G = \{g(x) \mid x \in A\}$, then

$$\sup\{f(x) + g(y) \mid x, y \in A\} = \sup F + \sup G$$

can be easily deduced since $F + G = \{f(x) + g(y) \mid x, y \in A\}$.

But $f + g \subseteq F + G$ so we of course get

$$\sup_A(f+g) \le \sup(F+G) = \sup\{f(x) + g(y) \mid x, y \in A\}$$

by monotonicity. But they may not be equal. Here is the idea. We want to construct two different functions f, g which attain their maximums at different points in A. For example, take $f, g : [0, \pi/2] \to \mathbb{R}$ given by

$$f(x) = \sin x, \quad g(x) = \cos x.$$

Then

$$(f+g)(x) = \sin x + \cos x = \sqrt{2}\left(\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x\right) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \le \sqrt{2}.$$

But $\sup_A f = 1 = \sup_A g$ and thus $\sup_A(f+g) = \sqrt{2} < 2 = \sup_A(f) + \sup_A(g)$.

5 Quiz 1

1. (a) (3pts) Let A be a nonempty subset of $\mathbb{R}$. State the definition of the infimum of A, in any of its equivalent forms.
   (b) (2pts) State the density of rationals.

Proof. (a) $m \in \mathbb{R}$ is an infimum of the set $A \subseteq \mathbb{R}$ if it satisfies: (1) m is a lower bound of A, *i.e.* for any $a \in A$, we have $m \leq a$, and (2) For any lower bound m' of A, $m' \leq m$, or equivalently, for any $m' > m$, there is an $a \in A$ such that $a < m'$, or equivalently, for any $\epsilon > 0$, there is an $a \in A$ such that $a < m + \epsilon$.
(b) That the set $\mathbb{Q}$ is dense in $\mathbb{R}$ means for any $a, b \in \mathbb{R}$ with $a < b$, there is a rational number $r \in \mathbb{Q}$ such that $a < r < b$. □

2. (5pts) Let $A = \left\{ 2 + \frac{1}{n} + \frac{1}{n^2} \middle| n \in \mathbb{N} \right\}$. Find $\inf A$ and prove your assertion.

Proof. First of all A is bounded from below, for instance by 2 since

$$2 < 2 + \frac{1}{n} + \frac{1}{n^2}$$

for all $n \in \mathbb{N}$. Thus $\inf A$ exists. Claim that $\inf A = 2$. For any $\epsilon > 0$, by the Archimedean Property, there exists a natural number $N \in \mathbb{N}$ such that $2/\epsilon < N$. Thus we get an element $2 + 1/N + 1/N^2 \in A$ such that

$$2 + \frac{1}{N} + \frac{1}{N^2} \leq 2 + \frac{1}{N} + \frac{1}{N} = 2 + \frac{2}{N} < 2 + \epsilon.$$

Therefore $\inf A = 2$. □

6 September 6

This week we start section about sequences. We say that a real sequence $\{x_n\}$ (which is defined to be a function $f : \mathbb{N} \to \mathbb{R}$ by assigning n to x_n) converges to a finite $L \in \mathbb{R}$ if for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ holds for all $n \geq N$. And we often write $\lim_{n\to\infty} a_n = L$ or $x_n \to L$ as $n \to \infty$. A sequence $\{x_n\}$ diverges if for any $L \in \mathbb{R}$ there is an $\epsilon > 0$ such that for any $N \in \mathbb{N}$ there is an $n \geq N$ such that $|x_n - L| \geq \epsilon$, though this is not the usual way to show some sequence is divergent. Beside, the notion of divergence means "not convergent", including infinite limits. while convergence only refers to finite limits. Note that we say "a limit of a sequence exists" if it is either finite or infinite. We will talk about them later.

Let's begin with some examples viewing limits of sequences by mainly using the definition.

Example 6.1. (1) $\lim_{n\to\infty}\left(3\left(1+\frac{1}{n}\right)\right)=3$. For any $\epsilon>0$, by Archimedean Property, there is an $N\in\mathbb{N}$ such that $3/\epsilon<N$. For any $n\geq N$, we have

$$\left|3\left(1+\frac{1}{n}\right)-3\right|=\frac{3}{n}\leq\frac{3}{N}<\epsilon.$$

(2) $\left\{\dfrac{1}{n}+\dfrac{\sin(n)}{n+1}\right\}\to 0$ as $n\to\infty$. First of all using the triangle inequality, we have

$$\left|\frac{1}{n}+\frac{\sin(n)}{n+1}\right|\leq\left|\frac{1}{n}\right|+\left|\frac{1}{n+1}\right| \text{(or simply } <2/n).$$

If using $2/n$ as a bound, then one can simply choose $N=[2/\epsilon]+1$. If using the sum as a bound, then for any $\epsilon>0$, there is an $N_1\in\mathbb{N}$ with $N_1\geq 2/\epsilon$ by A.P. such that $|1/n|\leq|1/N_1|<\epsilon/2$ for all $n\geq N_1$. There is also an $N_2\in\mathbb{N}$ with $N_2\geq 2/\epsilon-1$ by A.P. such that $|1/(n+1)|\leq|1/(N_2+1)|<\epsilon/2$. Then take $N=\max\{N_1,N_2\}\in\mathbb{N}$ we have

$$\left|\frac{1}{n}+\frac{\sin(n)}{n+1}\right|\leq\frac{1}{n}+\frac{1}{n+1}\leq\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

(3) $\left\{\dfrac{6n^2+5}{2n^2-3n}\right\}\to 3$ as $n\to\infty$. First of all, observe that

$$\left|\frac{6n^2+5}{2n^2-3n}\right|=\left|\frac{9n+5}{2n^2-3n}\right|<\frac{10n}{n^2}=\frac{10}{n}$$

where the last inequality holds when $n>5$ (since it is easy to check that $9n+5<10n$ iff $n>5$ and $2n^2-3n>n^2$ iff $n(n-3)>0$). Thus for any $\epsilon>0$, by A.P., there is an $N_1\in\mathbb{N}$ with $N_1>10/\epsilon$ or $N_1=[10/\epsilon]+1$ if you want to make it explicit. Take $N=\max\{6,N_1\}\in\mathbb{N}$, then for any $n\geq N$,

$$\left|\frac{6n^2+5}{2n^2-3n}\right|<\frac{10}{n}\leq\frac{10}{N}<\epsilon.$$

(4) $\{1+(-1)^n\}$ diverges. Suppose it converges to some finite real number $L\in\mathbb{R}$. For $\epsilon=1$, we can find $N\in\mathbb{N}$ such that for any $n\geq N$,

$$|1+(-1)^n-L|<1.$$

But if $n\geq N$ is odd, then

$$|1+(-1)^n-L|=|L|<1\iff -1<L<1;$$

if $n\geq N$ is even, then

$$|1+(-1)^n-L|=|2-L|<1\iff 1<L<3.$$

It leads to a contradiction. Thus $\{1+(-1)^n\}$ cannot be convergent.

(5) The following example requires a little more work. Test the conv/div of the sequence $\{\sin(n) \mid n \in \mathbb{N}\}$.
By way of contradiction, suppose that $\{\sin(n)\}$ converges to some finite $L \in \mathbb{R}$. Then $\sin(n-1) \to L$ and $\sin(n+1) \to L$ as $n \to \infty$. By the limit theorem, we have $\sin(n+1) - \sin(n-1) \to 0$ as $n \to \infty$. But

$$\sin(n+1) - \sin(n-1) = 2\sin(1)\cos(n).$$

Thus $2\sin(1)\cos(n) \to 0$ as $n \to \infty$. Since $2\sin(1)$ is a constant, we must have $\cos(n) \to 0$ as $n \to \infty$ and thus $\cos^2(n) \to 0$ as $n \to \infty$. By the identity $\sin^2(n) + \cos^2(n) = 1$, we have $\sin^2(n) \to 1$ as $n \to \infty$. Together with the assumption, it implies $L^2 = 1$. Now consider

$$\sin(n+1) = \sin(n)\cos(1) + \sin(1)\cos(n) \to L, \quad n \to \infty$$

where the second term approaches to 0 eventually and the first term approaches to $\cos(1)L$ eventually. It yields $\cos(1)L = L$ which implies $\cos(1) = 1$ which is a contradiction. Therefore $\{\sin(n)\}$ is not convergent.

7 September 8

Besides the algebraic property we have used multiple times in the above examples, we also have the comparison properties for sequences by comparing their tail terms. One of the theorems might shorten our find-the-limit procedure is the squeeze theorem. It tells us that if $a_n \leq b_n \leq c_n$ holds eventually for sequences $\{a_n\}, \{b_n\}$, and $\{c_n\}$, and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$ as well.

Here are some examples.

(0) Show $\lim \frac{n}{2^n} = 0$.
First we write $2^n = (1+1)^n$ with binomial expansion:

$$(1+1)^n = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

Then

$$2^n > \binom{n}{2} = \frac{n(n-1)}{2}$$

holds eventually. Now we have

$$0 \leq \frac{n}{2^n} \leq \frac{2}{n-1}$$

holds eventually. By squeeze theorem, we proved the limit is zero.

(1) Given $a \in \mathbb{R}$ with $|a| < 1$. Show $\lim a^n = \lim na^n = 0$.
We have seen the proof for the first limit during the lecture. If $a = 0$, then the result is obvious. Now assume $a \neq 0$ with $|a| < 1$. Since $1/|a| > 1$, we can write

$$\frac{1}{|a|} = 1 + h \text{ for some } h > 0.$$

So for any $n \in \mathbb{N}$,

$$\frac{1}{|a|^n} = (1+h)^n = 1 + \binom{n}{1}h + \binom{n}{2}h^2 + \cdots + h^n.$$

We will use the second summand for the first limit and the third summand for the second limit. In other words, we get

$$\frac{1}{|a|^n} > nh \implies 0 < |a|^n < \frac{1}{nh},$$

and

$$\frac{1}{|a|^n} > \frac{n(n-1)}{2}h^2 \implies 0 < \frac{1}{n|a|^n} < \frac{2}{(n-1)h^2}.$$

Since $1/nh \to 0$ and $2/(n-1)h^2 \to 0$ eventually, by the squeeze theorem, $\lim |a|^n = \lim n|a|^n = 0$ which is equivalently $\lim a^n = \lim na^n = 0$.

(2) $\{\sqrt[n]{n}\} \to 1$ as $n \to \infty$. First of all $\sqrt[n]{n} \geq 1$, so we can write $\sqrt[n]{n} = 1 + h_n$ for all $n \in \mathbb{N}$ with $h_n \geq 0$. Using the binomial expansion, for $n > 1$,

$$n = (1+h_n)^n \geq 1 + nh_n + \frac{n(n-1)}{2}h_n^2 > \frac{n(n-1)}{2}h_n^2,$$

and thus

$$h_n^2 < \frac{2}{n-1} \implies 0 < h_n < \sqrt{\frac{2}{n-1}}.$$

By the squeeze theorem, we proved $h_n \to 0$ as $n \to \infty$ which means $\sqrt[n]{n} - 1 \to 0$ as $n \to \infty$. Therefore $\sqrt[n]{n} \to 1$ as $n \to \infty$.

(3) Using the squeeze theorem, one can also deduce the following nice result. Suppose we have a nonempty subset $E \subset \mathbb{R}$. If E is bounded from above, then by the Completeness Axiom, $a := \sup E$ exists. We can actually construct a sequence $\{a_n\} \subset E$ fully contained in E with limit $a = \sup E$. Here is the construction. For each $n \in \mathbb{N}$, take $\epsilon = 1/n > 0$, there is an $a_n \in E$ such that

$$a - \frac{1}{n} < a_n \leq a.$$

Apply the squeeze theorem, we get $a_n \to a = \sup E$ eventually as desired.

8 Quiz 2

1. (5pts) State the definition of the limit of a sequence $\{a_n\} \subset \mathbb{R}$.

   Proof. A sequence $\{a_n\}$ in $\mathbb{R}$ converges to a finite $L \in \mathbb{R}$ if for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. □

2. (5pts) Prove using the definition of the limit (find $N(\epsilon)$ for every $\epsilon > 0$) that

$$\lim_{n\to\infty} \frac{n\cos n}{n^2+1} = 0.$$

   Proof. For any $\epsilon > 0$, by Archimedean Property, there is an $N \in \mathbb{N}$ such that $N > 1/\epsilon$. Then for all $n \geq N$, we have

$$\left|\frac{n\cos n}{n^2+1}\right| \leq \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

   which shows the limit equals to 0. □

9 September 13

We call a real sequence $\{a_n\} \subset \mathbb{R}$ diverges to ∞ if for any positive $M > 0$ there exists an $N \in \mathbb{N}$ such that $a_n > M$ for all $n \geq N$. Or simply, $a_n > M$ eventually for all $M > 0$. Likewise, we way a real sequence $\{a_n\}$ diverges to $-\infty$ if for any negative $M < 0$ there exists an $N \in \mathbb{N}$ such that $a_n < M$ for all $n \geq N$. Or simply $a_n < M$ eventually for all $M < 0$.

Here are some (non)examples of infinite limits:

Example 9.1.
- $n^2 \to \infty$ as $n \to \infty$. This is because $n^2 > M$ eventually for all $M > 0$, for example take $N > \sqrt{M}$.
- $\{a_n\} = \{1, 2, 1, 3, 1, 4, 1, 5, 1, 6, \cdots\}$ does NOT diverge to ∞. For example, if we take $M = 10$, no matter what value $N \in \mathbb{N}$ takes, the term 1 always occurs at every odd position.
- $\{\ln n\} \to \infty$ as $n \to \infty$. This is because $\ln x$ is increasing and $\ln n > \ln e^M = M$ eventually for every $M > 0$, for example take $N > e^M$.

Now let us consider a special type of sequences, monotonic sequences. Given a real sequence $\{a_n\}$, recall that
- $\{a_n\}$ is called eventually increasing if $a_m \leq a_n$ holds eventually for $m \leq n$;
- $\{a_n\}$ is called eventually decreasing if $a_m \leq a_n$ holds eventually for $m \geq n$.

For (eventually) monotonic sequences, convergence is closely related (equivalent) to boundedness. This is the monotonic convergence theorem.

Theorem 9.2. *Given a real sequence $\{a_n\}$.*

- *Suppose $\{a_n\}$ is eventually increasing. Then $\{a_n\}$ converges if and only if it is bounded from above;*
- *Suppose $\{a_n\}$ is eventually decreasing. Then $\{a_n\}$ converges if and only if it is bounded from below.*

Note that there are multiple ways to check monotonicity for a given sequence. For example, suppose $\{a_n\}$ is a real sequence with $a_n > 0$ for all n or eventually, to check $\{a_n\}$ is increasing is equivalent to check

- $a_n \leq a_{n+1}$ eventually;
- $a_{n+1}/a_n \geq 1$ eventually.

The above can also be easily reformulated to eventually decreasing sequences and sequences eventually with negative terms.

Here are some examples:

(1) $a_n = \dfrac{n}{2^n}$.

Notice that $\{a_n\}$ is a sequence with positive terms, and

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}}\frac{2^n}{n} = \frac{n+1}{2n} \leq \frac{n+n}{2n} = 1.$$

It implies that $\{a_n\}$ is decreasing. Moreover $\{a_n\}$ is bounded from below, and thus it converges.

(2) $a_n = \dfrac{3^n}{1+3^{2n}}$.

Notice that $a_n > 0$ holds for all $n \in \mathbb{N}$. Moreover

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{1+3^{2(n+1)}}\frac{1+3^{2n}}{3^n} = 3\frac{1+9^n}{1+9^{n+1}} < 3\frac{1+9^n}{3(1+9^n)} = 1$$

where the last inequality holds for all n since $1 + 9^{n+1} > 3 + 3 \cdot 9^n$ holds for all $n \in \mathbb{N}$. This show that $\{a_n\}$ is strictly decreasing. Since it is also bounded from below, it is convergent.

(3) $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!}$.

By direct calculation we get

$$a_{n+1} = \frac{(2n+1)!!}{2^{n+1}(n+1)!} = \frac{(2n-1)!!}{2^n n!}\frac{2n+1}{2(n+1)} = a_n \frac{2n+1}{2(n+1)} < a_n$$

holds for all $n \in \mathbb{N}$. Thus $\{a_n\}$ is decreasing and it is also bounded from below by zero, so it converges.

Sequences sometimes may not be given explicitly in terms of the index n, like Fibonacci sequence (even if we can write down its general form). For the sequences with term a_{n+1} defined by expression with a_n is called recursive sequences. Given a recursive formula together with the initial term a_1, one can write down each term in this sequence. But the convergence of such recursive sequence may or may not rely on the value of the initial element. Here are some examples.

(1) $a_{n+1} = a_n^2$ where $a_1 = \alpha$.
We first consider the case when $0 < \alpha < 1$. Then $a_n \in (0, 1)$ for all $n \in \mathbb{N}$. Moreover, $a_{n+1} = a_n^2 < a_n$ shows that $\{a_n\}$ is decreasing. And it is bounded from below by zero, so it converges with limit L satisfying $L = L^2$. So the limit L is either 0 or 1. But since the sequence is decreasing the limit cannot be 1 (make it rigorous by taking $\epsilon < 1 - \alpha$ for instance). Thus the limit is zero.
When $\alpha > 1$. Then $a_n > 1$ for all $n \in \mathbb{N}$. Moreover $1 < a_n < a_n^2 = a_{n+1}$ means that $\{a_n\}$ is increasing. If it converges then the limit has to be either one or zero but both are impossible (why?). Thus the sequence is unbounded from above and it doesn't converge.

(2) $a_{n+1} = \sqrt{2 + a_n}$ with $a_1 = \sqrt{2}$. One can use induction to show the sequence is strictly increasing and bounded from above.
Claim that $\{a_n\}$ is strictly increasing. First of all $a_2 = \sqrt{2 + \sqrt{2}} > \sqrt{2} = a_1$. And assume $a_{k+1} > a_k$. Then $a_{k+2} = \sqrt{2 + a_{k+1}} > \sqrt{2 + a_k} = a_{k+1}$, done. Now claim that it is bounded from above by 3. Firstly $a_1 = \sqrt{2} < 3$. Assume $a_k < 3$. Then $a_{k+1} = \sqrt{2 + a_k} < \sqrt{5} < 3$, done. We showed that $\{a_n\}$ is strictly increasing and bounded from above, thus it converges with limit L satisfying $L = \sqrt{2 + L}$. Therefore L is either 2 or -1. But $\{a_n\}$ is a positive sequence so the limit has to be 2.

10 September 15

Let us first finish the example from last time:

(3) $a_{n+1} = \sin(a_n)$ without condition on a_1.
Draw the graphs of $y = \sin(x)$ and $y = x$, notice that

- i) $-1 \leq a_n \leq 1$ for all $n \geq 2$. Thus $\{a_n\}$ is bounded.
- ii) In $[-1, 1]$, x and $\sin(x)$ will have the same sign, *i.e.* either both positive or both negative. It means that for $n \geq 2$, all a_n's share the same sign since $\sin(a_n) = a_{n+1}$ and a_n have the same sign.
- iii) If $x \geq 0$, then $\sin(x) \leq x$; If $x \leq 0$, then $\sin(x) \geq x$.

Now we consider two cases.

- i) If $a_2 = \sin(a_1) \in [0, 1]$. Then all a_n with $n \geq 2$ are nonnegative. Namely $0 \leq a_n \leq 1$ for all $n \geq 2$. Then $a_{n+1} = \sin(a_n) \leq a_n$, so $\{a_n\}$ is decreasing. And it converges to zero.
- ii) If $a_2 = \sin(a_1) \in [-1, 0]$. Then all a_n with $n \geq 2$ are not positive. Namely $-1 \leq a_n \leq 0$ for all $n \geq 2$. Then $a_{n+1} = \sin(a_n) \geq a_n$, ans thus $\{a_n\}$ is increasing. And it converges to zero also.

This is an example of recursive sequence with convergence property doesn't depend on the value of the initial element.

Proposition 10.1. *If $\{x_n\} \to x$ as $n \to \infty$, then*
(1) $\{\sin x_n\} \to \sin x$ as $n \to \infty$;

(2) $\{\cos x_n\} \to \cos x$ *as* $n \to \infty$.

Proof. For any $\epsilon > 0$, since $\{x_n\} \to x$ as $n \to \infty$, there exists an $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ whenever $n \geq N$. Then

$$\begin{aligned} |\sin x_n - \sin x| &= \left| 2\cos\left(\frac{x_n + x}{2}\right) \sin\left(\frac{x_n - x}{2}\right) \right| \\ &\leq 2 \left| \sin\left(\frac{x_n - x}{2}\right) \right| \\ &\leq 2 \left| \frac{x_n - x}{2} \right| < \epsilon. \end{aligned}$$

For cosine, one needs to use the formula:

$$\cos x_n - \cos x = -2\sin\left(\frac{x_n + x}{2}\right) \sin\left(\frac{x_n - x}{2}\right).$$

□

Finally let's end with two examples applying fundamental limits.

Example 10.2. Find the limits of the following sequences by applying some fundamental limits:

(1) $\left\{ \left(1 + \frac{2}{\sqrt{n}}\right)^{\sqrt{\frac{n^2+5}{n+3}}} \right\}$.

Since

$$\sqrt{\frac{n^2 + 5}{n + 3}} = \sqrt{\frac{n^2(1 + 5/n^2)}{n(1 + 3/n)}} = \sqrt{n}\sqrt{\frac{1 + 5/n^2}{1 + 3/n}}$$

we have

$$\left(1 + \frac{2}{\sqrt{n}}\right)^{\sqrt{\frac{n^2+5}{n+3}}} = \left(\left(1 + \frac{2}{\sqrt{n}}\right)^{\sqrt{n}}\right)^{\sqrt{\frac{1+5/n^2}{1+3/n}}} \to e^2$$

as $n \to \infty$.

(2) $\left\{ \frac{\pi^{6/n^2} - 1}{\sin(3/n^2)} \right\}$.

We rewrite

$$\frac{\pi^{6/n^2} - 1}{\sin(3/n^2)} = \frac{\pi^{6/n^2} - 1}{6/n^2} \frac{6/n^2}{\sin(3/n^2)} = 2\left(\frac{\pi^{6/n^2} - 1}{6/n^2}\right)\left(\frac{3/n^2}{\sin(3/n^2)}\right) \to 2\ln\pi$$

as $n \to \infty$.

11 Quiz 3

1. (a) (2pts) State the definition of an eventually decreasing sequence.
   (b) (3pts) State the Monotonic Convergence Theorem.

Proof. (a) We say a sequence $\{a_n\}$ eventually decreasing if there exists some $N \in \mathbb{N}$ such that $a_n \geq a_{n+1}$ holds for all $n \geq N$.

(b) Given a sequence $\{a_n\} \subset \mathbb{R}$. If $\{a_n\}$ is eventually increasing, then it is bounded from above if and only if it converges. If $\{a_n\}$ is eventually decreasing, then it is bounded from below if and only if it converges. $\square$

2. (5pts) Prove using the Monotonic Convergence Theorem that the sequence $\{a_n\} \subset \mathbb{R}$ given by

$$a_n = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2^2}\right) \cdots \left(1 - \frac{1}{2^n}\right)$$

converges.

Proof. First notice that $(1 - 1/2^n)$ is positive for each $n \in \mathbb{N}$. Thus $a_n > 0$ for all $n \in \mathbb{N}$, *i.e.* $\{a_n\}$ is bounded from below by zero. Moreover, for each $n \in \mathbb{N}$,

$$a_{n+1} = a_n \left(1 - \frac{1}{2^{n+1}}\right) \leq a_n$$

implies that $\{a_n\}$ is monotonically decreasing. By the Monotonic Convergence Theorem, the sequence $\{a_n\}$ converges. $\square$

12 September 20

Given a real sequence $\{a_n\}$. Let $n_1, n_2, \cdots$ be positive integers such that $n_k < n_{k+1}$ for each $k \in \mathbb{N}$ then the sequence $\{a_{n_k}\} = \{a_{n_1}, a_{n_2}, \cdots\}$ is called a subsequence of $\{a_n\}$. A subsequence of $\{a_n\}$ can also be characterized by a function $f : \mathbb{N} \to \mathbb{N}$ given by $k \mapsto n_k$, meaning that the k-th term in the subsequence is the n_k-th term in the original sequence. Moreover, based on how subsequence is constructed, $n_k \geq k$ always holds for all $k \in \mathbb{N}$.

One of the nice properties regarding subsequences is that a sequence $\{a_n\}$ converges to a finite limit a if and only if every subsequence of $\{a_n\}$ converges to a. This gives an easy and efficient way to check a given sequence is not convergent.

This also leads to the notion of limit points of a sequence. A sequence may not converge but it may have convergent subsequence. A real number $a \in \mathbb{R}$ is called a limit point of $\{a_n\}$ if there exists a subsequence of $\{a_n\}$ with limit a. So, one can show that a real sequence is convergent if and only if it is bounded and has exactly one limit point.

Here are some standard examples.

Example 12.1. (1) $a_n = 3\left(1 - \dfrac{1}{n}\right) + 2(-1)^n$.

Take subsequences $\{a_{2k}\}$ and $\{a_{2k+1}\}$. One approaches to 5 but the other approaches to 1. 1 and 5 are two limit points of $\{a_n\}$. And such sequence is not convergent.

(2) $a_n = \sin\left(\dfrac{\pi\sqrt[3]{n}}{2}\right)$.

Take subsequences $\{a_{(4k)^3}\}$ and $\{a_{(4k+1)^3}\}$. One approaches to 0 and the other approaches to 1. Thus the sequence is not convergent. What are the limit points of this sequence?

(3) Define

$$a_n = \begin{cases} n & \text{if } n \text{ is odd,} \\ 1/n & \text{if } n \text{ is even.} \end{cases}$$

Such sequence $\{a_n\}$ contains an unbounded subsequence $\{a_{2k+1}\}$ so it is not convergent. But $\{a_n\}$ does have one limit point which is zero.

(4) Consider $\{r_1, r_2, \cdots\}$ as an enumeration of rational numbers $\mathbb{Q}$. Then every element $a \in \mathbb{R}$ is a limit point of $\{r_n\}$. This is because for each $k \in \mathbb{N}$ the interval $(a, a + 1/k)$ contains infinitely many rational numbers. So we could construct a subsequence $\{r_{n_k}\}$ with $|r_{n_k} - a| < 1/k$ for all $k \in \mathbb{N}$.

Actually, one can show that every sequence in $\mathbb{R}$ has a monotonic subsequence. We can then easily conclude that every bounded sequence in $\mathbb{R}$ has a convergent subsequence (Bolzano-Weierstrass Theorem). The standard way of proving the Bolzano-Weierstrass Theorem is to apply the Nested Interval Property.

Make sure you know how to apply Archimedean Property to show for example

$$\bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right] = [0, 1].$$

In general, it is not easy to show a given sequence converges since one has to guess the limit first. But Cauchy sequences provides a way to avoid the guesswork.

A sequence $\{a_n\}$ in $\mathbb{R}$ is call a Cauchy sequence if for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $m, n \geq N$. A real sequence is Cauchy if and only if it converges (Cauchy Criterion).

Example 12.2. (1) Given a sequence $\{a_n\} \subset \mathbb{R}$ with property

$$|a_n - a_{n+1}| \leq r^n$$

for all $n \in \mathbb{N}$ and a fixed $0 < r < 1$.

Sequences with this property is convergent. We want to show such sequence is

Cauchy. Assume $m > n$ and consider

$$\begin{aligned}|a_n - a_m| &\le |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m| \\ &\le r^n + r^{n+1} + \cdots + r^{m-1} \\ &= r^n(1 + r + \cdots + r^{m-n-1}) \\ &= r^n \frac{1 - r^{m-n}}{1-r} = \frac{r^n - r^m}{1-r} \le \frac{r^n}{1-r}.\end{aligned}$$

It is easy to find $N \in \mathbb{N}$ such that $\frac{r^n}{1-r} < \epsilon$ for all $n \ge N$ since $r \in (0,1)$.

(2) Consider

$$a_n = \frac{\cos 1}{1 \cdot 2} + \frac{\cos 2}{2 \cdot 3} + \cdots + \frac{\cos n}{n \cdot (n+1)}.$$

Claim $\{a_n\}$ is Cauchy. Assume $m > n$,

$$\begin{aligned}|a_n - a_m| &\le \frac{|\cos(n+1)|}{(n+1)(n+2)} + \frac{|\cos(n+2)|}{(n+2)(n+3)} + \cdots + \frac{|\cos(m)|}{m(m+1)} \\ &\le \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{m(m+1)} \\ &= \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} + \cdots + \frac{1}{m} - \frac{1}{m+1} \\ &= \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n}.\end{aligned}$$

Then we can take for example $N = [1/\epsilon] + 1$.

Note that if a real sequence $\{a_n\}$ is Cauchy, then by definition

$$|a_n - a_{n+1}| < \epsilon$$

holds eventually for any $\epsilon > 0$. But the converse is NOT true.

Example 12.3. (1) Take $a_n = \sqrt{n}$. Then

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$$

as $n \to \infty$. But $\{a_n\}$ is not Cauchy since is it not even bounded.

(2) Take $a_n = \ln n$. It is not Cauchy since it is not bounded. But

$$\ln(n+1) - \ln n = \ln \frac{n+1}{n} \to \ln 1 = 0$$

as $n \to \infty$.

13 September 22

Showing a real sequence is Cauchy instead of proving convergence directly is often used when the limit of such sequence is not so easy to guess. Even though we can prove the convergence of the following sequence directly we prove it is Cauchy using the definition of Cauchy sequences just for practice.

Example 13.1. Consider the sequence $a_n = \frac{n^2}{n^2+1}$. For any $\epsilon > 0$, there exists some natural number $N \in \mathbb{N}$ such that $2/\epsilon < N$ by Archimedean Property. For any $n, m \geq N$ we have

$$\begin{aligned}
|a_n - a_m| &= \left| \frac{n^2}{n^2+1} - \frac{m^2}{m^2+1} \right| \\
&= \left| 1 - \frac{1}{n^2+1} - 1 + \frac{1}{m^2+1} \right| \\
&\leq \frac{1}{n^2+1} + \frac{1}{m^2+1} \\
&< \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \epsilon.
\end{aligned}$$

This shows that the sequence $\{a_n\}$ is Cauchy.

The following is also a nice property of subsequences. Roughly speaking, the subsequences $\{a_{2k}\}$ and $\{a_{2k+1}\}$ of even and odd terms respectively can give us enough information about the convergence of $\{a_n\}$.

Proposition 13.2. *Given a sequence $\{a_n\} \subset \mathbb{R}$. If both subsequences $\{a_{2k}\}$ and $\{a_{2k+1}\}$ converge to the same limit, say L, then $\{a_n\}$ converges to L.*

Proof. For any $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that for all even indices $2k \geq N_1$

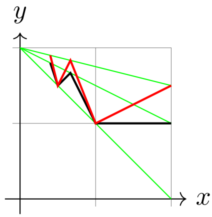
$$|a_{2k} - L| < \epsilon.$$

There also exists $N_2 \in \mathbb{N}$ such that for all odd indices $2k + 1 \geq N_2$

$$|a_{2k+1} - L| < \epsilon.$$

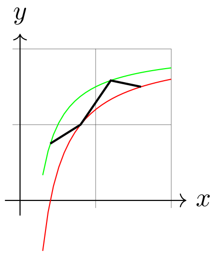
Now take $N = \max\{N_1, N_2\}$, then for any $n \geq N$, no matter whether n is even or odd, we always have $|a_n - L| < \epsilon$. It implies that $\{a_n\}$ converges to the limit L. $\square$

The above proposition provides an idea of constructing an example in Homework 2 Problem 4(b). We want to construct a real sequence $\{a_n\}$ with limit 1 and $a_n < 1$ for all $n \in \mathbb{N}$, but it is not eventually increasing. In other words, we want a convergent sequence "wiggling" towards 1 but strictly less than 1. The following two graphs gives a basic idea of construction.



For the first graph you could take your x values discretely as $1, 1/2, 1/3, \cdots, 1/n, \cdots$, and plug into two functions(lines) alternately given by $y = 1 - C_1 x$ and $y = 1 - C_2 x$ for some positive constants C_1, C_2. The constants C_i or the slopes of lines are manageable to guarantee the sequence to be not eventually increasing.

Or by taking reciprocal, or you don't like lines, you could also consider hyperbola in the following way:



For example, we can take two hyperbolas $y = 1 - C_1/x$ and $y = 1 - C_2/x$ and plug in $1, 2, \cdots, n, \cdots$ discretely and alternately, to get the sequence. The constants C_1, C_2 are still manageable to make the sequence to be not eventually increasing. Note that the hyperbolas you choose should have horizontal asymptote at $y = 1$ to ensure the limit is one. Two hyperbolas (or lines in the first graph) can be made to correspond to the even and odd subsequences, *i.e.* use $(-1)^n$ for example. Thus both these subsequences will share the same limit 1, by the proposition proved above, we get the limit of the sequence we constructed by jumping back and forth on two curves/lines is also 1. I will let you figure out how to make the example explicit.

Let's move on to the new chapter. Before doing the limits of functions, let's recall the notion of cluster points and isolated points of a subset in $\mathbb{R}$.

Given a subset $E \subseteq \mathbb{R}$. Then $x \in \mathbb{R}$ is

- an isolated point of E if $x \in E$ and there exists $\delta > 0$ such that $E \cap (x-\delta, x+\delta) = \{x\}$;
- a cluster point of E if for every $\delta > 0$ the interval $(x - \delta, x + \delta)$ contains a point in E which is distinct from x.

By definition, an isolated point of E must be in E. But a cluster point may not be an element in E.

Example 13.3.
- Consider the subset $E = (1,2) \cup (2,3) \cup \{4\}$.
  The set of cluster points of E, denoted by $E' = [1,3]$. And 4 is the only isolated point of E.
- Consider the subset $E = \{1/n \mid n \in \mathbb{N}\}$.
  Every point of E is an isolated point since one can always find a sufficient small interval about $1/n$ which does not contain $1/m$ for any $m \neq n$. The only cluster point of E is 0 which is not an element in E. Since every open interval about 0 contains $1/n$ for sufficiently large n.
- Consider $\mathbb{Q} \subset \mathbb{R}$.
  The set of rationals has no isolated point. And every real number is a cluster point of $\mathbb{Q}$.

14 Quiz 4

1. (1) (2pts) State the Bolzano-Weierstrass Theorem.
   (2) (3pts) State the definition of a Cauchy sequence $\{a_n\} \subset \mathbb{R}$.

Proof. (1) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.
(2) A real sequence $\{a_n\} \subset \mathbb{R}$ is Cauchy if for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for any $n, m \geq N$, $|a_n - a_m| < \epsilon$. $\square$

2. (5pts) Prove that the sequence $\{a_n\}$ given by

$$a_n = \frac{n \cdot (-1)^{\lfloor \frac{n}{3} \rfloor} + 2}{n+1}$$

is not convergent.

Proof. Take two subsequences $\{a_{6k}\}$ and $\{a_{3(2k+1)}\}$. Then

$$a_{6k} = \frac{6k(-1)^{2k} + 2}{6k+1} = \frac{6k+2}{6k+1} \to 1, \quad \text{as } k \to \infty;$$
$$a_{6k+3} = \frac{(6k+3)(-1)^{2k+1} + 2}{6k+3+1} = \frac{-(6k+3)+2}{6k+4} \to -1, \quad \text{as } k \to \infty.$$

Since $1 \neq -1$ the sequence $\{a_n\}$ cannot be convergent. $\square$

15 September 28

In the new chapter we will focus on real valued functions, Recall that a function is a subset $f \subset A \times B$ of the cartesian product such that for any $a \in A$ there is exactly one element $b \in B$ such that $(a,b) \in f$. We have introduced the concept of cluster point of a given subset of $\mathbb{R}$. Namely given a subset $A \subseteq \mathbb{R}$, we call $a \in \mathbb{R}$ a cluster point of the set A if *for any* $\epsilon > 0$ there is a point in $A \cap (a-\epsilon, a+\epsilon)$ other than a itself. Equivalently, if there is a sequence contained fully in $A \setminus \{a\}$ with limit equaling a. Or equivalently, for any $\epsilon > 0$ there is some $x \in A$ such that $0 < |x-a| < \epsilon$.

In this case, we will have sufficient amount of points near a to study the behavior of a function $f : A \to B$ when x approaching to such cluster point a.

Given a function $f : A \to \mathbb{R}$ with $a \in \mathbb{R}$ a cluster point of A (may not be a point of A). We have the following types of limit when $x \to a$, $x \to a^-$ and $x \to a^+$:

(1) We say a limit of $f(x)$ as x tends to a exists if there is a real number L such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $x \in A$ with $0 < |x-a| < \delta$ we have $|f(x) - L| < \epsilon$. Denote by $\lim_{x\to a} f(x) = L$.

(2) When restricting the above on $(-\infty, a) \cap A$ we get the definition of $\lim_{x\to a^-} f(x) = L$. Namely for any $\epsilon > 0$ there exists some $\delta > 0$ such that for any $x \in A$ with $a - \delta < x < a$ we have $|f(x) - L| < \epsilon$.

(3) Likewise we can define $\lim_{x\to a^+} f(x) = L$ by restricting on $(a, \infty) \cap A$.

Note that a limit is unique whenever it exists. And we also have the sequential characterization by saying that $\lim_{x\to a} f(x) = L$ if and only if for any sequence $\{x_n\} \subset A \setminus \{a\}$ we have $f(x_n) \to L$ as $n \to \infty$. The sequential argument is often used to show that the limit of a function does NOT exist.

We can also define the limit of f when x tends to $\pm\infty$ if $(c, \infty) \subset A$ or $(-\infty, c) \subset A$ for some $c \in \mathbb{R}$. Moreover, we can define infinite limits, in other words, define the function $f(x)$ converges to $\pm\infty$.

(4) $\lim_{x\to\infty} f(x) = L$ means for any $\epsilon > 0$ there exists an $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \epsilon$. One can define $\lim_{x\to-\infty} f(x) = L$ similarly.

(5) $\lim_{x\to a} f(x) = \infty$ means that for any $M \in \mathbb{R}$ there exists a $\delta > 0$ such that for any $x \in A$ with $0 < |x-a| < \delta$ we have $f(x) > M$. One can define $\lim_{x\to a} f(x) = -\infty$ similarly.

Her are some examples:

Example 15.1. • Consider a function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}; \\ x & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Check whether $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} f(x)$ exist or not.

First of all both 1 and 2 are cluster points of the domain $\mathbb{R}$. And for the limit as x tends to 1, we consider $|f(x) - 1|$ (with some guesswork by looking at the graph). We can write

$$|f(x) - 1| = \begin{cases} |x^2 - 1| = |x+1||x-1| & \text{if } x \in \mathbb{Q}; \\ |x-1| & \text{if } x \notin \mathbb{Q}. \end{cases}$$

If we consider $0 < |x-1| < 1$ (interval centered at 1 with radius 1), then $|x+1| = |x-1+2| \le |x-1| + 2 < 3$. In this case we have

$$|x^2 - 1| = |x+1||x-1| < 3|x-1|.$$

For any $\epsilon > 0$, one can take $\delta = \min\{1, \epsilon/3\}$ such that for any $x \in \mathbb{R}$ with $0 < |x-1| < \delta$, if $x \in \mathbb{Q}$, then $|f(x) - 1| = |x+1||x-1| < 3|x-1| < \epsilon/3 < \epsilon$; if $x \notin \mathbb{Q}$, we have $|f(x) - 1| = |x-1| < \epsilon$.
For the limit as x tends to 2, apply the density of $\mathbb{Q}$ and $\mathbb{Q}^c$ in $\mathbb{R}$. Namely there exist sequence $\{a_n\} \subset \mathbb{Q} \setminus \{2\}$ and $\{b_n\} \subset \mathbb{Q}^c$ with $a_n \to 2$ and $b_n \to 2$ as $n \to \infty$. But $f(a_n) = a_n^2 \to 4$ and $f(b_n) = b_n \to 2$ as $n \to \infty$ which are not equal. So by the sequential characterization the limit of f as x tends to 2 does NOT exist.

- Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^3 + 2x + 1.$$

Claim that $\lim_{x\to 1} f(x) = 4$. Consider

$$|f(x) - 4| = |x^3 + 2x - 3| = |x-1||x^2 + x + 3|.$$

Notice that when $|x-1| < 1$, *i.e.* $x \in (0, 2)$, since $x^2+x+3 = (x+1/2)^2+11/4$, this quadratic form is increasing on $(0, 2)$ and is always positive on $(0, 2)$. Thus $|x^2 + x + 3| < 9$ when $|x-1| < 1$. Then for any $\epsilon > 0$, one can take $\delta = \min\{1, \epsilon/9\}$. For any $0 < |x-1| < \delta$, we have

$$|f(x) - 4| = |x-1||x^2 + x + 3| < 9|x-1| < \epsilon.$$

- Prove the one-sided limit

$$\lim_{x\to 1^+} \frac{x-3}{3-x-2x^2} = \infty.$$

First of all note that

$$f(x) := \frac{x-3}{3-x-2x^2} = \frac{x-3}{(1-x)(3+2x)}.$$

Our goal is that for any $M > 0$ find $\delta > 0$ such that for any x with $1 < x < 1+\delta$, we have $f(x) > M$. For $1 < x < 1+\delta$, *i.e.* $-\delta < 1-x < 0$, one always have

$$\frac{1}{1-x} < -\frac{1}{\delta}.$$

For $1 < x < 1+1 = 2$, one have $-2 < x-3 < -1$ and $5 < 2x+3 < 7$. So

$$\frac{1}{7} < \frac{1}{2x+3} < \frac{1}{5}.$$

Take $\delta = \min\{1, 1/7M\}$, we have

$$\frac{x-3}{(1-x)(3+2x)} > \frac{x-3}{7(1-x)} > \frac{-1}{7(1-x)} > \frac{1}{7\delta} > M.$$

- Show that

$$\lim_{x\to\infty} \frac{\sin(x)}{x^2} = 0.$$

For any $\epsilon > 0$ take $M = 1/\sqrt{\epsilon}$ so that for any $x > M$ we have

$$\left|\frac{\sin(x)}{x^2}\right| \le \frac{1}{x^2} < \frac{1}{M^2} = \epsilon.$$

- Find the following limit

$$\lim_{x\to 0} x\cos\left(\frac{x^2+1}{x^3}\right).$$

First even if $x = 0$ is not in the domain of such function but it is a cluster point so we can still discuss its limit. Moreover we have

$$0 < \left|x\cos\frac{x^2+1}{x^3}\right| < |x|.$$

Since $|x| \to 0$ as $x \to 0$ by the squeeze theorem, we showed the limit is zero.

- Consider whether the following limit exist or not:

$$\lim_{x\to 1} \frac{1}{\ln x}.$$

Again $x = 1$ is not in the domain but it is a cluster point so it makes sense to talk about this limit. Take sequences $a_n = 1 + 1/n$ approaching to 1 as $n \to \infty$, and $b_n = 1 - 1/n$ also approaching to 1 as $n \to \infty$. But $1/\ln(a_n) \to \infty$ while $1/\ln(b_n) \to -\infty$ as $n \to \infty$ thus the limit does NOT exist by the sequential characterization.

16 October 4

We also have the corresponding algebraic properties of limits of functions. All of the followings can be shown by using the sequential argument but they can also be shown directly by definition.

Theorem 16.1. *Suppose that $f, g : A \to \mathbb{R}$ are functions and a is a cluster point of A. Assume that the limits*

$$\lim_{x\to a} f(x) = L, \quad \lim_{x\to a} g(x) = M$$

both exist. Then

$$\lim_{x\to a} kf(x) = kL, \quad \textit{for } k \in \mathbb{R};$$
$$\lim_{x\to a} [f(x) + g(x)] = L + M;$$
$$\lim_{x\to a} [f(x)g(x)] = LM;$$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad \textit{if } M \neq 0.$$

For algebraic operations on infinite limits, consider the following two cases and check whether true or not:

(i) If f, g are both *finite-valued* on some open interval $(a-1, a+1)$. Assume $\lim_{x\to a} f(x) = 0$, do we have $f(x)g(x) \to 0$ as $x \to a$?
"Finite-valued" here means that the function only takes values in $\mathbb{R}$. A finite-valued function may NOT be bounded. For example consider $f(x) = 1/x$, it is not bounded on its domain but it is finite-valued.
The statement is false since we can take $f(x) = x$ and $g(x)$ as

$$g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Consider $a = 0$. Note that $g(x)$ is unbounded but is finite-valued. And $\lim_{x\to 0} f(x) = 0$ but

$$f(x)g(x) = \begin{cases} 1 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Therefore $f(x)g(x) \to 1$ as x tends to 0.

(ii) If f, g are both *finite-valued* and g is *bounded* on the open interval $(a-1, a+1)$. Assume $\lim_{x\to a} f(x) = 0$, do we have $f(x)g(x) \to 0$ as $x \to a$?
If we put an extra bounded condition on g then the statement is true. Since g is bounded on $(a-1, a+1)$ there is $M > 0$ such that $|g(x)| \leq M$ whenever $x \in (a-1, a+1)$. Since $\lim_{x\to a} f(x) = 0$ for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $0 < |x-a| < \delta$ we have $|f(x)| < \epsilon/M$. Then

$$|f(x)g(x)| \leq M|f(x)| < \epsilon$$

whenever $0 < |x-a| < \delta$. It shows $\lim_{x\to a}[f(x)g(x)] = 0$ as desired.

In general we have the following properties for infinite limits. Assume $f, g : A \to \mathbb{R}$ and a is a cluster point of A:

- If $\lim_{x\to a} f(x) = L > 0$, and $\lim_{x\to a} g(x) = \infty$. Then $\lim_{x\to a} f(x)g(x) = \infty$. Give example that the result may fail if $L = 0$.

Proof. For any $M > 0$ we want to find $\delta > 0$ such that $x \in A$ and $0 < |x-a| < \delta$ imply $f(x)g(x) > M$. First there exists $\delta_1 > 0$ such that $f(x) > L/2$ for all $x \in A$(why?) and $0 < |x - a| < \delta_1$. There also exists $\delta_2 > 0$ such that $g(x) > 2M/L$ for all $x \in A$ and $0 < |x - a| < \delta_2$. Take $\delta = \min\{\delta_1, \delta_2\}$ then $x \in A$ with $0 < |x - a| < \delta$ implies

$$f(x)g(x) > \frac{L}{2}\frac{2M}{L} = M.$$

This proves the claim. But the conclusion may not hold if $L = 0$ for example take $f(x) = 1/x$ and $g(x) = x$ and consider $x \to \infty$. $\square$

- Suppose $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = L$ finite. Then $\lim_{x\to a}[f(x) - g(x)] = \infty$. If both limits are ∞, give example such that $\lim_{x\to a}[f(x) - g(x)] \neq \infty$.

Proof. For any $M > 0$. There exists $\delta_1 > 0$ such that $|g(x) - L| < 1$ whenever $x \in A$ and $0 < |x-a| < \delta_1$. Also there exists $\delta_2 > 0$ such that $f(x) > M+1+|L|$ whenever $x \in A$ and $0 < |x - a| < \delta_2$. Take $\delta = \min\{\delta_1, \delta_2\}$, then

$$f(x) - g(x) > M + 1 + |L| - 1 - L = M + |L| - L \geq M.$$

But the result may not hold if both have limit ∞. For example take $f(x) = 1+x$ and $g(x) = 2 + x$ and let $x \to \infty$. $\square$

N.B. There is one thing that I want to mention here is that given a sequence $\{a_n\}$, we can also view it as a set S consisting terms in $\{a_n\}$. We have seen the concept of a limit point of a sequence, namely, if a is a limit point of $\{a_n\}$ if there exists a subsequence with limit a. If view it is a set, we also define a cluster point of a set, which is a point on which every neighborhood contains a point in S other than a. It is not hard to show if a is a cluster point of S as a set, then it is also a limit point of $\{a_n\}$ as a sequence. But consider $a_n = (-1)^n$ we know ± 1 are limit points of sequence $\{a_n\}$ but as a set $S = \{-1, 1\}$ where ± 1 are isolated points not cluster points. So the converse may not be true.

Now let us move on to continuity of a real-valued function.

Given a function $f : A \to \mathbb{R}$ and $a \in A$. We say f is continuous at a if for any $\epsilon > 0$ there exists $\delta > 0$ such that $|x - a| < \delta$ and $x \in A$ imply $|f(x) - f(a)| < \epsilon$.

Remark 16.2.

- a must be a point in the domain A to define the continuity of f at a.
- (not so interesting) If $a \in A$ is an isolated point of A, then f is automatically continuous at a. (why?)
- In particular, if $a \in A$ is a cluster point of A, then f is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$.
- (Sequential) If $a \in A$ is a cluster point of A then f is continuous at a if and only if $\lim_{n\to\infty} f(a_n) = f(a)$ for every sequence $\{a_n\} \subseteq A$ with limit a.

17 October 6

Some examples:

Example 17.1. (1) Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We have seen in class that f is not continuous at 0 because the $\lim_{x\to 0} f(x)$ does not exist using sequential argument. Moreover it is continuous away from 0.

(2) Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

The function f is continuous on $\mathbb{R}$. We will only prove continuity at $x = 0$ for now. This is due to

$$|f(x) - f(0)| = |x \sin(1/x)| \leq |x|.$$

so $f(x) \to f(0)$ as $x \to 0$. What if $f(0)$ is defined to be any nonzero real number? Is the function still continuous at 0?

(3) Let $f : D \to \mathbb{R}$ where $D = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$. Then f is continuous at every $1/n$ for $n \in \mathbb{N}$ since each $1/n$ is an isolated point of D. If f is continuous at 0, then $f(1/n) \to f(0)$ by the sequential condition. If $f(1/n) \to f(0)$ as $n \to \infty$, then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f(1/n) - f(0)| < \epsilon$ whenever $n \geq N$. Take $\delta = 1/(N+1) > 0$. Then for any $x \in D$ with $|x| < \delta = 1/(N+1)$ means for all $x = 1/n$ for all $n \geq N+2$, we have

$$|f(x) - f(0)| < \epsilon.$$

It shows f is continuous at 0. Therefore f is continuous at 0 if and only if $f(1/n) \to f(0)$ as $n \to \infty$.

(4) Let $a \in \mathbb{R}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x > 0, \\ x + a & \text{if } x \leq 0. \end{cases}$$

For what value of the constant a is the function f continuous at 0?
Since $x = 0$ is a cluster point of the domain $\mathbb{R}$, f is continuous at $x = 0$ if and only if $\lim_{x\to 0} f(x) = f(0) = a$. $\lim_{x\to 0} f(x) = a$ requires

$$\lim_{x\to 0+} \sqrt{x} = a.$$

Thus $a = 0$ is the only value to make f continuous at 0.

Another nice problem: Consider a function $f : [0, \infty) \to \mathbb{R}$ which is assumed to be continuous, and with $\lim_{x\to\infty} f(x) = 1$. Then f is bounded over $[0, \infty)$. One cannot apply the extreme value theorem on $[0, \infty)$ since $[0, \infty)$ is not closed and bounded. But since $\lim_{x\to\infty} f(x) = 1$ there exists $M > 0$ such that for all $x > M$, $|f(x) - 1| < 1$ meaning that $|f(x)| \le |f(x) - 1| + 1 < 2$ on (M, ∞). Now we can apply the extreme value theorem on $[0, M]$ which is closed and bounded. There exists $K > 0$ such that $|f(x)| \le K$ for all $x \in [0, M]$. Thus for all $x \in [0, \infty)$, $|f(x)| \le \max\{2, K\}$ which shows f is bounded.

18 Quiz 5

1. (1) (2pts) Given a set $E \subseteq \mathbb{R}$. State the definition of $a \in \mathbb{R}$ being a cluster point of E.
   (2) (3pts) Given a function $f : E \to \mathbb{R}$ and a cluster point a of the domain $E \subseteq \mathbb{R}$. State the definition of the one-sided infinite limit

$$\lim_{x\to a^+} f(x) = -\infty.$$

Proof. (1) $a \in \mathbb{R}$ is a cluster point of the set E if for any $\delta > 0$, the intersection $(a - \delta, a + \delta) \cap (E \setminus \{a\})$ is nonempty.
(2) $\lim_{x\to a^+} f(x) = -\infty$ if for any $M < 0$ there is a $\delta > 0$ such that $f(x) < M$ whenever $a < x < a + \delta$. □

2. (5pts) Use the sequential characterization to prove that the limit

$$\lim_{x\to 1} \operatorname{sgn}\left(\sin\left(\frac{3}{x-1}\right)\right)$$

does not exist. Note that $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ is the sign function given by

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x > 0. \end{cases}$$

Proof. Take sequences

$$a_n = 1 + \frac{3}{2\pi n}, \quad b_n = 1 + \frac{3}{2\pi n + \frac{\pi}{2}}.$$

Both has limit 1 as $n \to \infty$. But

$$\operatorname{sgn}\left(\sin\left(\frac{3}{a_n - 1}\right)\right) = \operatorname{sgn}(\sin(2\pi n)) = 0,$$
$$\operatorname{sgn}\left(\sin\left(\frac{3}{b_n - 1}\right)\right) = \operatorname{sgn}(\sin(2\pi n + \pi/2)) = 1,$$

which yield two constant sequences with different limits. Therefore the limit does not exist. □

19 October 11

A continuous function has lots of nice properties. For example for a continuous function defined over a closed and compact interval, the max and min are always attained. Here is another one called intermediate value theorem which says that if $f : [a, b] \to \mathbb{R}$ is continuous on the closed and bounded interval, then f attains every value between maximum and minimum. In other words, the image $f([a, b]) = [m, M]$ is also a closed bounded interval. It is easy to construct examples to see both continuity on f and connectedness together with the compactness of the domain of f cannot be dropped.

We say a function f has the intermediate value property (IVP) on an interval I if for any $a, b \in I$ with $a < b$ and $z \in \mathbb{R}$ is any value between $f(a)$ and $f(b)$ then there exists $c \in [a, b]$ such that $f(c) = z$. Then every continuous function $f : I \to \mathbb{R}$ has IVP on I. But the converse is not true. Consider $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$ as an example.

Here are some applications of the intermediate value theorem:

Example 19.1.

- Consider the polynomial function

$$f(x) = x^7 + 4x^5 + 3x^4 - 6x^2 + 3,$$

  then such function has at least one real root. This is due to the fact that we can write

$$f(x) = x^7(1 + 4/x^2 + 3/x^3 - 6/x^5 + 3/x^7)$$

  then

$$\lim_{x\to\infty} f(x) = \infty, \quad \lim_{x\to-\infty} f(x) = -\infty.$$

  It means that we can find some $a, b \in \mathbb{R}$ with $a > 0$ and $b < 0$ such that $f(a) > 0$ and $f(b) < 0$. Now since f is continuous on $[b, a]$ we can apply intermediate value theorem on $[b, a]$ to get the existence of $c \in (b, a)$ such that $f(c) = 0$ (because $f(b) < 0 < f(a)$).
- Consider another polynomial function

$$f(x) = -x^4 + 2x^3 + 2$$

  then such function has at least two real roots.
  One way to see this is by checking $f(0) = 2 > 0$, $f(-1) = -1 < 0$ and $f(3) = -81 + 54 + 2 < 0$ and apply the intermediate value theorem on $[-1, 0]$ and $[0, 3]$ to get two different real roots.
  Or since $f(0) = 2 > 0$ and $\lim_{x\to\infty} f(x) = -\infty$ with $\lim_{x\to-\infty} f(x) = -\infty$, we will get the same result by apply intermediate value theorem on two intervals.

- Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Given $c_1, \cdots, c_n \in [a, b]$. Then there exists some $c \in [a, b]$ such that

$$f(c) = \frac{f(c_1) + \cdots + f(c_n)}{n}.$$

  Let j be such that $f(c_j)$ is the maximum among all $f(c_i)$'s; let k be such that $f(c_k)$ is the minimum among all $f(c_i)$'s. Then

$$f(c_k) \leq \frac{f(c_1) + \cdots + f(c_n)}{n} \leq f(c_j).$$

  And we can apply the intermediate value theorem on the interval with endpoints c_j and c_k.

Let us move on to the uniform continuous functions. A function $f : D \to \mathbb{R}$ is said to be uniformly continuous on D if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in D$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

Unlike the continuity of a function, for a uniformly continuous function, the choice of δ does not rely on the point $x \in D$.

Remark 19.2. Given a function $f : D \to \mathbb{R}$:
- If f is uniformly continuous on D then it is continuous on D.
- We also have the sequential characterization of uniformly continuous functions. f is uniformly continuous on D if and only if for any $\{x_n\}, \{y_n\} \subset D$ with $\lim_{n\to\infty} |x_n - y_n| = 0$ then $\lim_{n\to\infty} |f(x_n) - f(y_n)| = 0$.
  f is not uniformly continuous on D if and only if there exists some $\epsilon_0 > 0$ and $\{x_n\}, \{y_n\} \subset D$ such that $\lim_{n\to\infty} |x_n - y_n| = 0$ but $|f(x_n) - f(y_n)| > \epsilon_0$ for all $n \in \mathbb{N}$.
- Uniformly continuous function sends a Cauchy sequence to a Cauchy sequence.

Some examples:

Example 19.3.
- The function $f(x) = x^3 - x + 2$ is uniformly continuous on $(0, 1)$. By definition, for any $\epsilon > 0$, take $\delta = \epsilon/4$. Then for any $a, b \in (0, 1)$ with $|a - b| < \delta$:

$$\begin{aligned}
|f(a) - f(b)| &= |a^3 - a + 2 - b^3 + b - 2| \\
&= |a^3 - b^3 - (a - b)| \\
&= |a - b||a^2 + ab + b^2 - 1| \\
&\leq 4|a - b| < 4\delta = \epsilon.
\end{aligned}$$

- The function $f(x) = \sin(x^2)$ is uniformly continuous on $[a, b]$ but it is not uniformly continuous on $[a, \infty)$.

Let us first prove f is uniformly continuous on $[a,b]$. Take $M = \max\{|a|,|b|\}$. For any $\epsilon > 0$, take $\delta = \epsilon/2M$. Then for any $x,y \in [a,b]$ with $|x-y| < \delta$, we have

$$\begin{aligned}
|f(x) - f(y)| &= |\sin(x^2) - \sin(y^2)| \\
&= 2\left|\sin\frac{x^2+y^2}{2}\right|\left|\sin\frac{x^2-y^2}{2}\right| \\
&\leq 2\left|\sin\frac{x^2-y^2}{2}\right| \\
&\leq |x-y||x+y| \leq 2M|x-y| < \epsilon.
\end{aligned}$$

But the function f is not uniformly continuous on $[a,\infty)$. Take

$$x_n = \sqrt{n\pi}, \quad y_n = \sqrt{n\pi + \pi/2}.$$

Note that the sequences $\{x_n\},\{y_n\}$ are eventually in $[a,\infty)$. They also satisfy:

$$y_n - x_n = \frac{\pi/2}{\sqrt{n\pi}+\sqrt{n\pi+\pi/2}} < \frac{2}{2\sqrt{n}} = \frac{1}{\sqrt{n}} \to 0$$

as $n \to \infty$. But

$$|f(x_n) - f(y_n)| = 1$$

implies that f is not uniformly continuous on $[a,\infty)$.

- The function $f(x) = \sqrt{x}$ is also uniformly continuous on $[0,\infty]$. This is due to the fact that for any $x,y \geq 0$:

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}.$$

- As a generalization of above example, we call a function $f : D \to \mathbb{R}$ satisfying

$$|f(x) - f(y)| \leq C|x-y|^\alpha$$

for all $x,y \in D$, Hölder continuous of order α where C is a nonnegative real constant and $\alpha > 0$. In particular, if $\alpha = 1$ then we say the function satisfies the Lipschitz condition. A function is Hölder continuous implies uniformly continuous.

20 October 13

The continuous extension theorem allows one to define the value of the function at the endpoints of the domain to extend continuous function to a (uniformly) continuous function over the closed and bounded domain.

Example 20.1. Consider the function

$$f(x) = x^\alpha \sin\left(\frac{1}{x}\right)$$

for some constant power $\alpha \in \mathbb{R}$ on the open interval $(0,1)$.

If $\alpha > 0$ then

$$\left| x^\alpha \sin\left(\frac{1}{x}\right) \right| \leq |x|^\alpha \to 0$$

as $x \to 0^+$. Thus $f(x)$ is uniformly continuous on $(0,1)$ for all $\alpha > 0$.

If $\alpha \leq 0$ then take a sequence

$$x_n = \frac{2}{(2n+1)\pi}.$$

But $f(x_n) = (-1)^n x_n^\alpha$ does not converge as $n \to \infty$. Thus the function is not uniformly continuous on $(0,1)$ for $\alpha \leq 0$.

We also have the similar property for infinite domain:

Proposition 20.2. *Suppose $f : [0,\infty) \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x) = L < \infty$. Then f is uniformly continuous on $[0,\infty)$.*

Proof. Since $f(x) \to L$ as $x \to \infty$, for any $\epsilon > 0$ there exists $M > 0$ such that for any $x \geq M$ we have

$$|f(x) - L| < \epsilon/3.$$

Since $f : [0,\infty) \to \mathbb{R}$ is continuous, f is UC on $[0, M]$. Thus there exists $\delta > 0$ such that for any $x, y \in [0, M]$ with $|x - y| < \delta$ we have

$$|f(x) - f(y)| < \epsilon/3.$$

Now let $x, y \in [0,\infty)$, and suppose $|x - y| < \delta$:

- If $x, y \in [0, M]$, we are done.
- If $x, y \in (M,\infty)$, then

$$|f(x) - f(y)| \leq |f(x) - L| + |f(y) - L| < 2\epsilon/3 < \epsilon.$$

- If $x \in [0, M]$ and $y \in (M,\infty)$, then $|x - M| \leq |x - y| < \delta$, thus

$$|f(x) - f(y)| \leq |f(x) - f(M)| + |f(M) - L| + |f(y) - L| < \epsilon.$$

Therefore f is UC on $[0,\infty)$. □

Example 20.3. By above proposition, the function $f(x) = \frac{1}{1+x^2}$ is UC on $\mathbb{R}$.

21 Quiz 6

1. Suppose $f : D \to \mathbb{R}$ is a function defined on $D \subseteq \mathbb{R}$.
   (1) (2pts) State the definition of f being continuous at $a \in D$.

   Proof. The function f is continuous at $a \in D$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $x \in D$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$. □

   (2) (3pts) State the definition of f being uniformly continuous on D.

   Proof. The function f is uniformly continuous on D if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $x, y \in D$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$. □

2. Let $f : [0, 2] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1; \\ (2 - x)(x^5 + 1) & \text{if } 1 \le x \le 2. \end{cases}$$

   (1) (2pts) Is the function continuous on $[0, 2]$? Justify your answer.

   Proof. No. The function is not continuous at $x = 1$ because

$$\lim_{x \to 1^-} f(x) = 1 \neq 2 = \lim_{x \to 1^+} f(x).$$ □

   (2) (3pts) Prove f attains every value between 0 and 2.

   Proof. The function is clearly continuous on $[1, 2]$ since it is a polynomial function. Since $f(1) = 2$ and $f(2) = 0$, by applying the Intermediate Value Theorem to f on $[1, 2]$, the function must attain all values between 0 and 2. □

22 October 18

A function that is differentiable at a point means that it can be locally approximated by a linear function with the slope of tangent line given by the derivative. It is also a local property meaning that the differentiablity and the value of the derivative depend only on the values of function f in an arbitrary small neighborhood of such point. Namely, we say the function $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$ with derivative $f'(c)$ if the limit exists and is finite

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = f'(c).$$

And the domain of f' is the set of points c for which the above limit exists. This definition is usually used when c is an interior point of the domain of the function so that one can always find a δ-neighborhood of c entirely contained in the domain.

There is also another nice and convenient way to characterize differentiability called Carathéodory Lemma:

Lemma 22.1 (Carathéodory Lemma)**.** *Given a function $f : D \to \mathbb{R}$. The function f is differentiable at an interior point $a \in D$ if and only if there exists a increment function $\mathcal{D} : D \to \mathbb{R}$ such that*

$$f(x) - f(a) = \mathcal{D}(x)(x - a)$$

for any $x \in D$ and $\mathcal{D}$ is continuous at a. In this case we have $\mathcal{D}(a) = f'(a)$.

One can check that if such $\mathcal{D}(x)$ exists it is uniquely determined by f and a. Moreover the continuity of $\mathcal{D}$ at a implies the continuity of f at a.

Now let's look at some examples.

Example 22.2.

- Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational;} \\ -x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

  It is easy to check that f is only continuous at $x = 0$. Thus f is not differentiable at any $a \in \mathbb{R}$ with $a \neq 0$. Now we define

$$\mathcal{D}(x) = \begin{cases} x & \text{if } x \text{ is rational;} \\ -x & \text{if } x \text{ is irrational.} \end{cases}$$

  It is continuous at $x = 0$ and $f(x) = x\mathcal{D}(x) + f(0)$ holds for all $x \in \mathbb{R}$. So by the Carathéodory Lemma, f is differentiable at $x = 0$.
- Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = |x|^{1/2}.$$

  If $a > 0$, then

$$\begin{aligned} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{(a+h)^{1/2} - a^{1/2}}{h} \\ &= \lim_{h \to 0} \frac{h}{h[(a+h)^{1/2} + a^{1/2}]} \\ &= \lim_{h \to 0} \frac{1}{(a+h)^{1/2} + a^{1/2}} \\ &= \frac{1}{2a^{1/2}}. \end{aligned}$$

  For negative $a < 0$, the limit can be computed similarly with a negative sign. But f is not differentiable at $x = 0$ this is due to the limit

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{1}{h^{1/2}}$$

does not exist as a finite limit. Therefore f is differentiable at $x \neq 0$ with derivative:

$$f'(x) = \frac{\operatorname{sgn}(x)}{2|x|^{1/2}}.$$

- Now consider the function $f : (0, \infty) \to \mathbb{R}$ given by

$$f(x) = x^\alpha$$

for some fixed power $\alpha \in \mathbb{R}$. Then for any $x \in (0, \infty)$,

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(x+h)^\alpha - x^\alpha}{h} \\
&= \lim_{h \to 0} \left(\frac{\left(1 + \frac{h}{x}\right)^\alpha - 1}{\frac{h}{x}} \right) (x^{\alpha-1}) \\
&= \alpha x^{\alpha-1}.
\end{aligned}$$

Therefore the function f is differentiable on $(0, \infty)$ with derivative $f'(x) = \alpha x^{\alpha-1}$.
One could then ask for what values of α is the function $f(x) = x^\alpha$ continuous at 0.

The followings are some highly oscillating functions:

Example 22.3.

- Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

It is easy to see that f is differentiable at $x \neq 0$ with derivative (compute by yourself). But since the limit

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin(1/h)}{h} = \lim_{h \to 0} \sin(1/h)$$

does not exist, the function f is not differentiable at $x = 0$.
- Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Again it is easy to check that the function f is differentiable at $x \neq 0$. Moreover

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h \sin(1/h) = 0$$

thus the function is differentiable on $\mathbb{R}$. Check yourself that $f'(x)$ is not continuous at $x = 0$.

23 October 20

The extreme value theorem for continuous functions tells us the existence of absolute max/min on a closed and bounded domain but it doesn't say how to find them. One of an important applications of differentiability is to locate the max/min of functions. I will skip the definitions of global/local extrema, check the lecture notes if needed.

Let us first recall some definitions. Given a function $f : D \to \mathbb{R}$:

- A point $c \in D$ is called a critical point of f if c is an interior point of D such that either f is not differentiable at c or $f'(c) = 0$;
- A point $c \in \mathbb{R}$ is called a boundary point of D if there exist sequences $\{x_n\} \subset D$ and $\{y_n\} \subset \mathbb{R} \setminus D$ both of limit c. Equivalently, $c \in \mathbb{R}$ is a boundary point of D if and only if for any $\delta > 0$ the neighborhood $(c - \delta, c + \delta)$ contains a point of D as well as a point not in D.

Theorem 23.1 (Fermat)**.** *If $f : D \to \mathbb{R}$ has a local extremum at an interior point $c \in D$ and f is differentiable at c, then $f'(c) = 0$.*

In particular, if the domain $D \subset \mathbb{R}$ is closed and bounded and $f : D \to \mathbb{R}$ is continuous, then absolute max and min exist. And the absolute max and min are attained at either a critical point of f or a boundary point of D.

Remark 23.2. In Fermat's theorem, the condition on c as an interior point is crucial since we need to compare the signs of left/right difference quotients at c. But at an endpoint of an interval domain, we can also get similar results. For instance, let $f : [a, b] \to \mathbb{R}$.

- Suppose the left derivative of f exists at b. If f has a local max at b then $f'(b-) \geq 0$;
- Suppose the right derivative of f exists at a. If f has a local max at a then $f'(a+) \leq 0$.

Complete the statement for local min yourself. It is easier to see by drawing graphs.

Let's look at some examples. Note that there is no standard way to search local max and min.

Example 23.3.

- Consider the floor function $f(x) = \lfloor x \rfloor$ for $x \in \mathbb{R}$. Then every integer is a local max. Every element in $\mathbb{R} \setminus \mathbb{Z}$ is both a local min and max. There is no global max or min for f on $\mathbb{R}$.
- (A critical point may not be local extrema) Consider $f : [-1, 1] \to \mathbb{R}$:

$$f(x) = \begin{cases} x & \text{if } x \in [-1, 0]; \\ 2x & \text{if } x \in (0, 1]. \end{cases}$$

  Then f is not differentiable at only $x = 0$ and $x = 0$ is the unique critical point of f on $[-1, 1]$. But f does not attain local extrema at $x = 0$. Indeed, the absolute max and min of f attains at two endpoints. And f has no other local extrema. (There are also examples such as $f(x) = x^3$ which has been discussed in lectures.)

- Now consider $f(x) = x^4 - 2x^2 - 1$ on $[-2, 2]$. Then the origin $x = 0$ is a local max. This is simply because $f(0) = -1$ and

$$f(0) - f(x) = 2x^2 - x^4 = x^2(2 - x^2) \geq 0$$

holds on $[-1, 1]$. But it is not global because $f(2) = 16-8-1 = 7 > f(0) = -1$.

- Consider the function

$$f(x) = x + \frac{1}{1+x}$$

on $(-1, \infty)$. Then $x = 0$ is a local min and also a global min. This is because

$$f(x) - f(0) = x + \frac{1}{1+x} - 1 = \frac{x^2}{1+x} \geq 0$$

on $(-1, \infty)$.

24 Quiz 7

1. (5 pts) State the definition of a function $f : (a, b) \to \mathbb{R}$ being differentiable at $c \in (a, b)$.

Proof. We say the function $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$ if the limit

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists and is finite. We then denote the limit by $f'(c)$ which is called the derivative of f at c. □

2. (5 pts) Let $f : (0, \infty) \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \in (0, 1); \\ -x + 3 & \text{if } x \in [1, \infty). \end{cases}$$

(1) (3 pts) Use the definition to prove the function f is differentiable on the interval $(0, 1)$.

Proof. For any $x \in (0, 1)$, since the limit

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \to 0} \frac{-1}{x(x+h)} \\ &= -\frac{1}{x^2}. \end{aligned}$$

the function is differentiable on $(0, 1)$ with derivative given by $f'(x) = -1/x^2$. $\square$

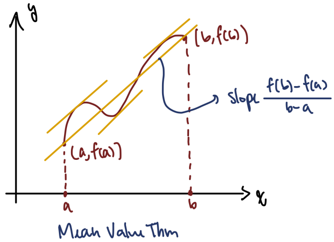
(2) (2 pts) Is the function differentiable on $(0, \infty)$? Justify your answer.

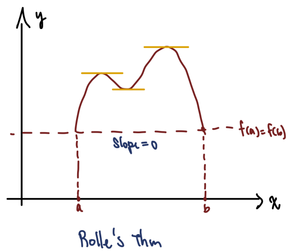
Proof. No. The function f is not continuous at $x = 1$ due to

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{x} = 1 \neq 2 = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-x + 3). \qquad \square$$

25 October 25

The Mean Value Theorem tells us that for a function $f : [a, b] \to \mathbb{R}$ which is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ at which the tangent line is parallel to the line joining two endpoints $(a, f(a))$ and $(b, f(b))$. Rolle's Theorem is the special case of MVT when the line joining two endpoints is horizontal. The Mean Value Theorem is incredibly useful which can be used to prove Fundamental Theorem of Calculus, L'Hopital's Rule and etc. Geometrically, we can interpret MVT as follows:





Mean Value Theorem has a lot of applications. We have seen one in class that suppose $f : [a, b) \to \mathbb{R}$ is continuous on $[a, b)$ and differentiable on the open (a, b), if the one-sided limit $\lim_{x \to a^+} f'(x)$ of derivative exists (finite/infinite), then the sided derivative also exists with the same value, *i.e.*

$$f'_+(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^+} f'(x).$$

We use that to define a plane curve (as graph of some function) has left/right vertical tangent line at certain point. Here are some examples of curves with a cusp:

Example 25.1. • Consider the function $f(x) = x^{2/3}$ on $\mathbb{R}$ and the origin $x = 0$. We know this function is continuous on $\mathbb{R}$. Moreover,

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{1}{\sqrt[3]{x}} = \infty,$$

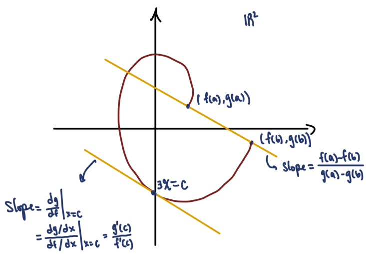
while

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x}=\lim_{x\to 0^-}\frac{1}{\sqrt[3]{x}}=-\infty.$$

Thus f is not differentiable at $x=0$ and it has a cusp at $x=0$.

- Other example you may want to check yourself is $f(x)=\sqrt{|x|}$ at $x=0$.

The MVT can also be used to prove a general version. With two functions $f,g:[a,b]\to\mathbb{R}$ being continuous on $[a,b]$ and differentiable on (a,b) we also have the generalized MVT, with geometric interpretation as follow:



"Cauchy MVT"

Other applications are

(0) (Uniqueness of roots) Consider $f(x)=x^4+2x^3-2$ for $x\in\mathbb{R}$. Since $f'(x)=4x^3+6x^2>0$ for all $x\in(0,\infty)$ by Rolle's Theorem the function f has at most one root in $[0,\infty)$. On the other hand, $f(0)=-2$ and $f(1)=1$ by IVT, f has at least one root in $[0,1]$. Therefore f has a unique root in $[0,\infty)$.

(1) (Estimate values) For instance we want to estimate $\sqrt{2}$. Consider $f(x)=\sqrt{x}$ and apply MVT to $f(x)$ on $[1,2]$. Then there exists $c\in(1,2)$ such that

$$f'(c)=\frac{1}{2\sqrt{c}}=\frac{\sqrt{2}-1}{1}.$$

Since $c\in(1,2)$ we have

$$\frac{1}{2\sqrt{2}}\le f'(c)=\sqrt{2}-1<\frac{1}{2}.$$

Thus we obtain

$$1+\frac{1}{2\sqrt{2}}<\sqrt{2}<\frac{3}{2}.$$

Use the second inequality we have $\frac{1}{\sqrt{2}}>\frac{2}{3}$, thus $\frac{4}{3}<\sqrt{2}<\frac{3}{2}$.

For some similar exercises, one can try proving using MVT:

$$\frac{17}{9}<\sqrt[3]{7}<\frac{23}{12},\quad \frac{1}{9}<\sqrt{66}-8<\frac{1}{8}.$$

(2) (Prove inequalities)
   (i) Show $-a \le \sin(a) \le a$ for all $a \ge 0$.
   If $a = 0$ we have $\sin(0) = 0$. If $a > 0$, consider $f(x) = \sin(x)$. Then f is clearly continuous on $[0, a]$ and differentiable on $(0, a)$. Apply MVT on $[0, a]$, there exists $c \in (0, a)$ such that $f(a) - f(0) = \cos(c)a$ which implies that $-a \le f(a) = \sin(a) \le a$ since $\cos(c) \in [-1, 1]$.
   (ii) (Bernoulli) Show if $\alpha > 1$, then $(1 + x)^\alpha \ge 1 + \alpha x$ for all $x > -1$.
   Let $f(x) = (1 + x)^\alpha$ then $f'(x) = \alpha(1 + x)^{\alpha-1}$ for all $x > -1$. Consider two cases when $x > 0$ and $-1 < x < 0$ (it is trivial when $x = 0$). If $x > 0$ apply MVT to $f(x)$ on the closed interval $[0, x]$. Then there exists some $c \in (0, x)$ such that

$$f(x) - f(0) = f'(c)x \implies (1 + x)^\alpha - 1 = \alpha(1 + c)^{\alpha-1}x.$$

   Since $c > 0$ we have $(1 + c)^{\alpha-1} > 1$. Thus

$$(1 + x)^\alpha - 1 > \alpha x.$$

   If $-1 < x < 0$ then apply MVT to f on $[x, 0]$, we will get the same result.

(3) (Prove uniform continuity) Consider the function $f(x) = \log x$ we want to show that it is uniformly continuous on $(1, \infty)$.
For any $x, y \in (1, \infty)$ assume $x < y$. Apply MVT to $f(x)$ on $[x, y]$. Then there exists $c \in (x, y)$ such that

$$f(y) - f(x) = f'(c)(y - x) \implies \log(y) - \log(x) = \frac{1}{c}(y - x).$$

Taking absolute value, we get

$$|f(x) - f(y)| = \frac{1}{c}|x - y| < |x - y|$$

since $c > 1$. Therefore f is Lipschitz on $(1, \infty)$ thus it is uniformly continuous on $(1, \infty)$.

26 October 27

Here is a sufficient condition for a local extremum.

Proposition 26.1. *Let $D \subseteq \mathbb{R}$ and c is an interior point of D, and $f : D \to \mathbb{R}$. Assume f is continuous at c.*

- *(First derivative test for local minimum) f is differentiable on $(c-\delta, c) \cup (c, c+\delta)$ for some $\delta > 0$. And $f'(x) \le 0$ for all $x \in (c - \delta, c)$ while $f'(x) \ge 0$ for all $x \in (c, c + \delta)$. Then f has a local minimum at c.*

- *(First derivative test for local maximum) f is differentiable on $(c-\delta, c) \cup (c, c+\delta)$ for some $\delta > 0$. And $f'(x) \geq 0$ for all $x \in (c-\delta, c)$ while $f'(x) \leq 0$ for all $x \in (c, c+\delta)$. Then f has a local maximum at c.*

Proof. For the first case we know that f is decreasing on $(c-\delta, c)$ and increasing on $(c, c+\delta)$. Moreover since f is continuous at c by taking the limits as $x \to c^-$ and $x \to c^+$ we see that $f(x) \geq f(c)$ for all $x \in (c-\delta, c+\delta)$. Thus f has a local minimum at c. The proof for the local maximum follows similarly. □

Example 26.2. Consider $f : (-1, 1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 < |x| < 1, \\ -1 & \text{if } x = 0. \end{cases}$$

It is clear that f has a (strict) local minimum at $x = 0$. But f is not continuous at 0 so the conditions in the first derivative test are not satisfied. Also we can assign $f(0) = 1$. Then f is differentiable on $(-1, 0) \cup (0, 1)$ and $f'(x) \leq 0$ on $(-1, 0)$ and $f'(x) \geq 0$ on $(0, 1)$. But f doesn't have a local minimum at 0.

Note that f is not required to be differentiable at point c. Here is an example:

Example 26.3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^x + x - 1 & \text{if } x > 0; \\ -3x^3 - x & \text{if } x \leq 0. \end{cases}$$

Then it can be easily checked that f is continuous everywhere on $\mathbb{R}$. And $f'_-(0) = -1 \neq 2 = f'_+(0)$ so f is not differentiable at $x = 0$. But f still has a local minimum at $x = 0$.

Also, it is possible to have a function which is differentiable on $\mathbb{R}$ and has absolute minimum at $x = 0$. But its derivative is not ≥ 0 in any interval $(0, \delta)$.

Example 26.4. Consider function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 2x^2 + x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It is easy to see that $f(x) > 0 = f(0)$ for any $x \neq 0$. Thus f has an absolute minimum at $x = 0$. And it can also be check easily that f is differentiable at $x = 0$. To see f' is not ≥ 0 in any interval $(0, \delta)$, we consider the derivative

$$f'(x) = \begin{cases} 4x + 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x} & \text{if } x \neq 0, \\ f'(0) = 0 & \text{if } x = 0. \end{cases}$$

Since

$$f'\left(\frac{1}{2n\pi}\right) = \frac{2}{n\pi} + 0 - 1 < 0, \quad \text{for all } n \in \mathbb{N}$$

and

$$f'\left(\frac{1}{2n\pi + \pi/2}\right) = \frac{4}{2n\pi + \pi/2} + \frac{2}{2\pi n + \pi/2} > 0, \quad \text{for all } n \in \mathbb{N},$$

for any $\delta > 0$, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{2n\pi + \pi/2} < \frac{1}{2n\pi} < \delta.$$

Thus in any interval $(0, \delta)$, f' can take both positive and negative values.

Similarly one can consider function

$$f(x) = \begin{cases} x + 2x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

And show that f is differentiable on $\mathbb{R}$ and in every neighborhood of 0, the derivative f' takes both positive and negative values.

27 Quiz 8

1. (4 pts) State the Mean Value Theorem.

   Proof. Suppose a function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable in (a, b) then there exists a point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. □

2. (3 pts) Prove the following inequality using Mean Value theorem

   $$e^x > 1 + x, \quad \text{for all } x > 0.$$

   (Hint: Consider for example, the function $f(x) = e^x - x - 1$.)

   Proof. Define $f(x) = e^x - x - 1$ which is clearly continuous and differentiable on $\mathbb{R}$. Fix any $a > 0$, apply the Mean Value Theorem to f on the interval $[0, a]$. Then there exists some $c \in (0, a)$ such that $f(a) - f(0) = f'(c)a$. Since $f'(x) = e^x - 1$ we can rewrite the equation as

   $$e^a - a - 1 = (e^c - 1)a > 0$$

   since $e^c - 1 > e^0 - 1 = 0$ for $c > 0$. Therefore $e^a - a - 1 > 0$ for any $a > 0$ which shows $e^x > 1 + x$ for all $x > 0$.

   □

3. (3 pts) Is it possible that a function f is continuous on $[1,3]$, differentiable in $(1,3)$, $f(1)=4$, $f(3)=0$ and $|f'(x)|<2$ for every $x\in(1,3)$? Justify your answer.

Proof. No. Since f is continuous on $[1,3]$ and differentiable in $(1,3)$ by Mean Value Theorem, there exists a point $c\in(1,3)$ such that

$$f'(c)=\frac{f(3)-f(1)}{2}=\frac{-4}{2}=-2.$$

Thus there is a $c\in(1,3)$ with $|f'(c)|=2$. □

28 November 1

For today's recitation we are going to continue on L'Hopital's Rule and take a look at a "discrete" analogue of L'Hopital's rule on sequences which might be used as an alternatively way to show the L'Hopital's for ∞/∞ type of limits sequentially, and turns out to be useful in finite difference equations with a byproduct of computing the limits of some types of series.

Before getting to the Toepliz-Stolz-Cesàro Theorem, let's first have a quick review on when the L'Hopital's rule can be applied. In order to apply the L'Hopital's rule to compute

$$\lim_{x\to a^+}\frac{f(x)}{g(x)}$$

for $\lim_{x\to a^+}f(x)=\lim_{x\to a^+}g(x)=0$ (for instance). One needs to check that

- f and g are differentiable on some (a,b) with $g'(x)\neq 0$ on (a,b) – $g'(x)$ did not change sign infinitely often in a (right) neighborhood of a;
- $\lim_{x\to a^+}\frac{f'(x)}{g'(x)}=L$ exists finitely or infinitely.

Then we can conclude that the target limit exists and equals to L.

Let's consider several (non)examples regarding the L'Hopital's Rule:

Example 28.1. (1) Compute

$$\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}.$$

This limit can be easily computed using some basic algebra. But if we apply L'Hopital blindly we make ourselves in the infinite loop. L'Hopital's Rule is of course a incredibly useful technique but do not forget other methods of computing limits.

(2) Consider a modification of an example from class:

$$\lim_{x\to\infty}\frac{x-\sin x}{x+\sin x}=1.$$

We have both $f(x) = x - \sin x$ and $g(x) = x + \sin x$ are differentiable on $\mathbb{R}$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. But L'Hopital cannot be applied because

$$g'(x) = 1 + \cos x$$

and $g'(x) = 0$ for $x = 2n\pi + \pi$ for any $n \in \mathbb{Z}$. It means that $g'(x)$ has zeros in every interval (M, ∞). Therefore the L'Hopital's rule cannot be applied here.

(3) Let's first briefly define the second derivative here. Let $f : D \to \mathbb{R}$ and suppose f is differentiable on the subset $E \subseteq D$. Let $c \in E$ be a cluster point of E. If $f' : E \to \mathbb{R}$ is differentiable at c then we denote $f''(c) = (f')'(c)$ and call it the second derivative of f at c.
Now consider a function $f(x)$ which is defined in a neighborhood of c. And suppose $f''(c)$ exists. Then

$$\lim_{h\to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2} = f''(c).$$

But even if the above limit exists $f''(c)$ does not necessarily exist.
Since $f''(c)$ exists, by definition we have

$$f''(x) = \lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h\to 0} \frac{f'(c) - f'(c-h)}{h}.$$

Then

$$\begin{aligned} f''(x) &= \frac{1}{2} \lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} + \frac{1}{2} \lim_{h\to 0} \frac{f'(c) - f'(c-h)}{h} \\ &= \lim_{h\to 0} \frac{f'(c+h) - f'(c-h)}{2h} \end{aligned}$$

On the other hand, applying L'Hopital and chain rule we have

$$\lim_{h\to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2} = \lim_{h\to 0} \frac{f'(c+h) - f'(c-h)}{2h}.$$

Thus

$$f''(c) = \lim_{h\to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2}.$$

For the counterexample, we want to construct a function f with $f'(x) = |x|$, thus f'' does not exist at $x = 0$. So we consider $f(x) = \int_0^x |t| \mathrm{d}t = x|x|$ with $c = 0$. Then again by L'Hopital, the limit is 0 but the $f''(0)$ does not exist.

Now let's move to the Toeplitz-Stolz-Cesàro. Let me first state the Toeplitz Theorem without proof:

Theorem 28.2 (Toeplitz). *Let $\{c_{n,k} \mid 1 \leq k \leq n, n \in \mathbb{N}\}$ be a double-indexed set of points in $\mathbb{R}$,* i.e.

$$\begin{array}{cccc} c_{1,1} & & & \\ c_{2,1} & c_{2,2} & & \\ \vdots & \vdots & & \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

such that

- *each vertical sequence converges to zero,* i.e. *for each fixed $k \geq 1$, the sequence $c_{n,k} \to 0$ as $n \to \infty$;*
- *the sequence formed by horizontal sums $\sum_{k=1}^{n} c_{n,k} \to 1$ as $n \to \infty$;*
- *there exists a positive constant $C > 0$ such that $\sum_{k=1}^{n} |c_{n,k}| \leq C$ for each $n \in \mathbb{N}$.*

Then for any convergent sequence $\{a_n\}$ the sequence $\{b_n\}$ given by

$$b_n = \sum_{k=1}^{n} c_{n,k} a_k$$

is also convergent with $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

Note that the proof of the Toeplitz Theorem is not hard one can first reduce to assuming the sequence $\{a_n\}$ has limit zero (why?).

The Toeplitz Theorem produces fruitful results. I will only name a few here:

(a) Take a special array by setting $c_{n,k} = \frac{1}{n}$ for all $1 \leq k \leq n$. Namely the nth row of the above table will be $(\frac{1}{n}, \cdots, \frac{1}{n})$ and the columns are some truncations of the sequence $\{\frac{1}{n}\}$. It can be easily checked that such $\{c_{n,k}\}$ satisfies the conditions in the Toeplitz Theorem. Thus we get the following proposition:

Proposition 28.3. *Suppose $\{a_n\} \to L$ converges and $\{b_n\}$ is defined by*

$$b_n = \frac{a_1 + \cdots + a_n}{n}.$$

Then $\{b_n\} \to L$ as $n \to \infty$.

Given a sequence $\{a_n\}$ and define $\{b_n\}$ as above, *i.e.* taking the arithmetic means of $\{a_n\}$, We say $\{a_n\}$ is **Cesàro convergent** if $\{b_n\}$ is convergent. By the above proposition, we know if $\{a_n\}$ is convergent then it must be Cesàro convergent. But the converse may not be true, for example take $a_n = (-1)^n$ (check).

We can use the above result to compute the limit of some series, for example

$$\left\{ b_n = \frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \right\}.$$

Since we know that $a_n = \frac{1}{n} \to 0$ then $b_n \to 0$ as well by the proposition.

(b) The Toeplitz Theorem also implies the Stolz Theorem:

Theorem 28.4 (Stolz)**.** *Let $\{a_n\}, \{b_n\}$ be two sequences such that*

i) $\{b_n\}$ *strictly increasing to* ∞.

ii) $\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = L.$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = L.$

The above Stolz Theorem might be used to give a sequential proof of L'Hopital's Rule for ∞/∞ type of limits. Moreover it can also be used to compute the limit of some sequences:

1) Consider
$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right).$$
Take $a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ and $b_n = \sqrt{n}$ which is strictly increasing to infinity. Since
$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = 2,$$
by Stolz Theorem, the target limit is also 2.

2) (Exercise) Find the limit
$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}, \quad k \in \mathbb{N}.$$
using the Stolz Theorem.

29 November 3

We have already seen the definition of higher order derivatives in yesterday's lecture. Roughly speaking, given a function, in order to define the nth order derivative we just consider the $(n-1)$th derivative on the subset where it is differentiable at. As mentioned in class, the domains may not always behave nicely, *i.e.* such as intervals or union of intervals, it might be some set consisting of sequences as well.

Let's consider an example:

Example 29.1. Let $f : [0, 1) \to \mathbb{R}$ be a function defined by
$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \in (0,1) \cap \mathbb{Q}; \\ x^2 & \text{if } x \in (0,1) \cap \mathbb{Q}^c \text{ or } x = 0. \end{cases}$$
In order to find the set on which the function is differentiable, call it $D_1 \subseteq [0, 1)$, first notice that it has to be continuous on such set. Thus
$$D_1 \subseteq \left\{ \frac{1}{2\pi n + \pi/2} \,\middle|\, n \in \mathbb{N} \right\} \cup \{0\}$$

since the function is continuous only at zero and the point $x \in (0,1)$ with $\sin(1/x) = 1$. This can be argued as following. Take any sequence $\{a_k\} \to x_n$, then $\{a_k\}$ can be divided into three cases: contains infinitely many rationals and irrationals; only infinitely rationals; only infinitely many irrationals. No matter what situation, together with

$$\lim_{x \to x_n} x^2 \sin\frac{1}{x} = x_n^2 = \lim_{x \to x_n} x^2$$

we obtain the continuity of f at x_n.

One can also check that the function f is also differentiable at these x_n's together with the origin. At $x = x_n$, consider

$$\lim_{h \to 0} \frac{f(x_n + h) - f(x_n)}{h} = 2x_n$$

which can be argued similarly as above. At the origin $x = 0$, we simply have

$$\left|\frac{f(h)}{h} - 0\right| = \begin{cases} \left|\dfrac{h^2 \sin\frac{1}{h}}{h}\right| & \text{if } h \in \mathbb{Q}; \\ |h| & \text{if } h \notin \mathbb{Q}. \end{cases}$$

in any event, the above value $\leq |h|$ which has limit 0 as $h \to 0^+$. Therefore f is right-differentiable at 0 with $f'_+(0) = 0$ and differentiable at x_n's with $f'(x_n) = 2x_n$. Thus

$$D_1 = \left\{ \frac{1}{2\pi n + \pi/2} \,\middle|\, n \in \mathbb{N} \right\} \cup \{0\}$$

is the domain for the first order derivative f'. The only accumulation point in D_1 is 0. One can also check if f' is right-differentiable at 0. Since

$$\lim_{n \to \infty} \frac{f'(x_n) - f'(0)}{x_n - 0} = \lim_{n \to \infty} \frac{2x_n}{x_n} = 2,$$

it means that the function f has second (right) derivative at 0 with value $f''(0) = 2$.

Besides the above extreme example, let us consider the following:

Example 29.2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

It is clear that the function f is differentiable away from $x = 0$. At $x = 0$, we have

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-\frac{1}{h^2}}}{h} = \lim_{h \to 0} \frac{1/h}{e^{\frac{1}{h^2}}} = \lim_{h \to 0} \frac{h}{2e^{\frac{1}{h^2}}} = 0$$

by L'Hopital's Rule. Thus

$$f'(x) = \begin{cases} \dfrac{2}{x^3} e^{-\frac{1}{x^2}} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

It can be checked similarly that f' is continuous on $\mathbb{R}$. Moreover, we have

$$f''(x) = \begin{cases} e^{-\frac{1}{x^2}} \left(\dfrac{4}{x^6} - \dfrac{6}{x^4} \right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Because using L'Hopital we have

$$\lim_{h \to 0} \frac{f'(h) - f'(0)}{h} = \lim_{h \to 0} \frac{2e^{-\frac{1}{h^2}}}{h^4} = 0.$$

Consequently, we get for any $n \in \mathbb{N}$

$$f^{(n)}(x) = \begin{cases} e^{-\frac{1}{x^2}} P(1/x) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0, \end{cases}$$

where $P(x)$ is a polynomial in $\mathbb{R}[x]$. Therefore $f \in C^\infty(\mathbb{R})$. The nth Taylor's polynomial of f at zero can be written as

$$T_n(x; 0; f) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = 0.$$

30 Quiz 9

1. (4 pts) State the characterization of local minimum using first derivatives.

   Proof. Let $D \subseteq \mathbb{R}$ and c be an interior point of D. Suppose $f : D \to \mathbb{R}$ is continuous at c. If f is differentiable on $(c - \delta, c) \cup (c, c + \delta)$ for some $\delta > 0$ and $f'(x) \leq 0$ for all $x \in (c - \delta, c)$ while $f'(x) \geq 0$ for all $x \in (c, c + \delta)$, then f has a local minimum at c. □

2. (6 pts) Consider the function $f : (0, \infty) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } 0 < x < 1; \\ 0 & \text{if } x = 1; \\ x^2 - 2\ln x & \text{if } x > 1. \end{cases}$$

(a) (2 pts) Find an interval on which the function f is decreasing. Justify your answer.

Proof. On $(0,1)$ it is easy to see that $f'(x) = -x^{-2} < 0$. Thus f is decreasing on $(0,1)$. $\square$

(b) (2 pts) Find an interval on which the function f is increasing. Justify your answer.

Proof. On $(1,\infty)$ we have $f'(x) = 2x - \frac{2}{x} = \frac{2x^2-2}{x} = \frac{2(x-1)(x+1)}{x} > 0$. Thus the function f is increasing in any interval contained in $(1,\infty)$. $\square$

(c) (2 pts) Does the function f attain its local minimum at $x = 1$? Justify your answer.

Proof. We know from part (a) and (b) that the function f is decreasing in $(0,1)$ and increasing in $(1,\infty)$. Moreover

$$\lim_{x\to 1^+} f(x) = 1 = \lim_{x\to 1^-} f(x) > f(1) = 0.$$

Therefore for any $x \in (0,\infty)\setminus\{1\}$ we have $f(x) \geq 1 > f(1) = 0$ which means that the function has an absolute min at $x = 1$. $\square$

31 November 15 & 17

Before doing examples regarding Taylor's formula, let's first finish up several examples about application of the Mean Value Theorem:

Example 31.1. (1) Give an example of a differentiable function f for which $\lim_{x\to\infty} f(x)$ exists but $\lim_{x\to\infty} f'(x)$ does not exist.
Consider for example

$$f(x) = \frac{1}{x}\sin(x^2).$$

It is easy to see that $\lim_{x\to\infty} f(x) = 0$ but since

$$f'(x) = 2\cos x^2 - \frac{\sin x^2}{x^2}$$

the limit $\lim_{x\to\infty} f'(x)$ does not exist.

(2) Show that if $\lim_{x\to\infty} f(x) < \infty$ and $\lim_{x\to\infty} f'(x)$ exists then $\lim_{x\to\infty} f'(x) = 0$.
Let $L = \lim_{x\to\infty} f'(x)$. Assume $|L| > 0$. Then there exists some $M > 0$ such that whenever $x \geq M$ we have $|f'(x) - L| < |L|/2$. Thus we have for all $x \geq M$, $|f'(x)| > |L|/2$. But then for any $x > M$ apply Mean Value Theorem on $[x, x+1]$ there exists $c \in (x, x+1)$ such that

$$|f(x+1) - f(x)| = |f'(c)| > |L|/2$$

which violates $\lim_{x\to\infty} f(x)$ exists and is finite.

Let us then move on to the Taylor's formula. Recall the Taylor formula (with remainder in Lagrange form) says: If $n \in \mathbb{N}$ and $f^{(k)}$'s exist for all $n = 1, \cdots, n$ and are continuous on some $[a, b]$. In other words, f is of class $C^n([a, b])$. Assume $f^{(n+1)}$ exists on (a, b). If $x_0 \in [a, b]$ then for any $x \in [a, b]$ there exists some c between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

The sum of first $n+1$ terms is called the nth Taylor's polynomial of f at x_0, denoted by $T_n(x; x_0; f)$; the last term is called the nth Taylor's remainder in Lagrange form, denoted by $R_n(x; x_0; f)$.

There are tons of applications regarding to Taylor's formula, let's take a look at some of them.

Example 31.2. Use Taylor's formula to do estimation.

(1) We have seen example estimating $\sqrt{2}$ using Mean Value Theorem. The result gives

$$\frac{4}{3} < \sqrt{2} < \frac{3}{2}.$$

We could get a better estimation of $\sqrt{2}$ by using Taylor's formula. Again let $f(x) = \sqrt{x}$ and we will only use the first Taylor polynomial to see if we could get better bounds. Use Taylor's formula with $n = 1$, for any $x > 1$ we get

$$f(x) = f(1) + f'(1)(x - 1) + \frac{f''(c)}{2}(x - 1)^2$$

for some $c \in (1, x)$. We know $f'(x) = \frac{1}{2\sqrt{x}}$ and $f''(x) = -\frac{1}{4\sqrt{x^3}}$, then take $x = 2$ we have

$$\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{8\sqrt{c^3}}.$$

Since $c \in (1, 2)$ we have $\frac{1}{\sqrt{c^3}} < 1$ so

$$\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{8\sqrt{c^3}} > 1 + \frac{1}{2} - \frac{1}{8} = \frac{11}{8} > \frac{4}{3}.$$

Moreover we also have $\frac{1}{\sqrt{c^3}} > \frac{1}{\sqrt{2^3}} = \frac{1}{2\sqrt{2}}$ so

$$\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{8\sqrt{c^3}} < 1 + \frac{1}{2} - \frac{1}{16\sqrt{2}} < \frac{35}{24} < \frac{3}{2}.$$

Therefore we get a slightly better estimation for $\sqrt{2}$:

$$\frac{11}{8} < \sqrt{2} < \frac{35}{24}.$$

(2) Using the similar idea by considering $f(x) = \ln(1+x)$ and expanding with some truncated Taylor's polynomial together with the remainder, one can also estimate $\ln(2)$ up to any accuracy. Claim that for $0 \le x \le 1$ (it is true for $-1 < x \le 1$ but the estimation for the remainder for $-1 < x < 0$ requires using either integral form of Cauchy form):

$$f(x) = \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Since $f^{(n+1)}(x) = (-1)^{n+2} n!(1+x)^{-(n+1)}$ we can write the nth remainder as

$$|R_n(x;0;f)| = \frac{|x|^{n+1}}{|(n+1)(1+c)^{n+1}|}$$

for some $c \in (0,x)$. Since $0 \le x \le 1$ we can estimate

$$|R_n(x;0;f)| \le \frac{1}{(n+1)} \to 0$$

as $n \to \infty$. Therefore $f(x) = \ln(1+x)$ can be written in an infinite sum when $x \in [0,1]$. In particular we have

$$\ln(2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}.$$

If we use degree 3 Taylor's polynomial we get

$$f(1) = \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4(1+c)^4}$$

for some $c \in (0,1)$. Thus we get $\ln 2$ can be approximated by $1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ with error

$$\left| \frac{1}{4(1+c)^4} \right| < \frac{1}{4}.$$

In general we could get an inequality for $x > 0$:

$$\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2n-1}}{2n-1},$$
$$\ln(1+x) > x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n}.$$

(3) Use the above inequality on $\ln(1+x)$ we can compute for example the following limit:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2}{n^2}\right) \cdots \left(1 + \frac{n}{n^2}\right).$$

Let us first transform from product to sum by taking log. Denote

$$a_n := \left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right),$$

and then

$$\ln a_n = \sum_{k=1}^{n} \ln\left(1+\frac{k}{n^2}\right).$$

By the inequality we have above

$$\frac{k}{n^2} - \frac{k^2}{2n^4} < \ln\left(1+\frac{k}{n^2}\right) < \frac{k}{n^2}.$$

Taking summation from 1 to n:

$$\frac{1}{n^2}\sum_{k=1}^{n} k - \frac{1}{2n^4}\sum_{k=1}^{n} k^2 < \ln a_n < \frac{1}{n^2}\sum_{k=1}^{n} k$$

which can be further simplified as

$$\frac{n(n+1)}{2n^2} - \frac{n(n+1)(2n+1)}{6\cdot 2n^4} < \ln a_n < \frac{n(n+1)}{2n^2}.$$

By squeeze theorem we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^{1/2}.$$

Example 31.3. Other than estimation of some exact number, it can be also used to find bounds in an abstract way.

(1) Given a function f. Suppose f', f'' both exist on $[0,1]$. If $|f(0)| \le 1$, $|f(1)| \le 1$ and moreover $|f''(x)| \le 2$ for any $x \in [0,1]$. Then what can we say about $|f'(x)|$ on $(0,1)$?
Fix any $x \in (0,1)$ by Taylor's formula at x we have

$$f(1) = f(x) + f'(x)(1-x) + \frac{f''(c)}{2}(1-x)^2$$

for some $c \in (x,1)$ and

$$f(0) = f(x) + f'(x)(0-x) + \frac{f''(d)}{2}(0-x)^2$$

for some $d \in (0,x)$. Therefore we get

$$f(1) - f(0) = f'(x) + \frac{f''(c)}{2}(1-x)^2 - \frac{f''(d)}{2}x^2$$

which implies

$$\begin{aligned}
|f'(x)| &= \left| f(1) - f(0) - \frac{f''(c)}{2}(1-x)^2 + \frac{f''(d)}{2}x^2 \right| \\
&\leq |f(1)| + |f(0)| + \frac{1}{2}|f''(c)|(1-x)^2 + \frac{1}{2}|f''(d)|x^2 \\
&\leq 2 + (1-x)^2 + x^2 \leq 2 + 1 = 3.
\end{aligned}$$

So we get $|f'(x)|$ is bounded from above by 3 on $(0, 1)$.

(2) Given a function f. Suppose f', f'', f''' all exist on $\mathbb{R}$. Moreover assume $f(x), f'''(x)$ are bounded on $\mathbb{R}$. Then is $f'(x)$ bounded on $\mathbb{R}$? How about $f''(x)$?

Fix any $x \in \mathbb{R}$. By Taylor's formula of f at x we can write

$$f(x+1) = f(x) + f'(x) + \frac{f''(x)}{2} + \frac{f'''(c)}{3!}$$

for some $c \in (x, x+1)$ and similarly,

$$f(x-1) = f(x) + f'(x)(-1) + \frac{f''(x)}{2}(-1)^2 + \frac{f'''(d)}{3!}(-1)^3$$

for some $d \in (x-1, x)$. Thus

$$f(x+1) - f(x-1) = 2f'(x) + \frac{1}{6}[f'''(c) + f'''(d)].$$

Therefore, we obtain an expression of $|f'(x)|$ as

$$\begin{aligned}
|f'(x)| &= \frac{1}{2}\left| f(x+1) - f(x-1) - \frac{1}{6}[f'''(c) + f'''(d)] \right| \\
&\leq \frac{1}{2}\left(|f(x+1)| + |f(x-1)| + \frac{1}{6}|f'''(c)| + \frac{1}{6}|f'''(d)| \right) \\
&\leq M_0 + \frac{1}{6}M_3
\end{aligned}$$

where M_0 and M_3 are bounds for $|f(x)|$ and $|f'''(x)|$ on $\mathbb{R}$ respectively. Therefore we have showed that f' is also bounded on $\mathbb{R}$.

For $f''(x)$ on $\mathbb{R}$, summing $f(x+1)$ and $f(x-1)$ up yields,

$$f(x+1) + f(x-1) = 2f(x) + f''(x) + \frac{1}{6}[f'''(c) - f'''(d)].$$

Likewise we have an expression for $|f''(x)|$ now:

$$\begin{aligned}
|f''(x)| &= \left| f(x+1) + f(x-1) - 2f(x) - \frac{1}{6}[f'''(c) - f'''(d)] \right| \\
&\leq 4M_0 + \frac{1}{3}M_3,
\end{aligned}$$

which means that $f''(x)$ is also bounded on $\mathbb{R}$.

(3) (Exercise) Suppose the function $f(x)$ is second differentiable on $\mathbb{R}$. Let

$$M_k := \sup_{x\in\mathbb{R}} |f^{(k)}(x)|, \quad k = 0, 1, 2.$$

If $M_k < \infty$ for $k = 0, 2$, prove $M_1^2 \leq 2M_0M_2$ and thus $M_1 < \infty$.

We have seen how Riemann integration is defined for a bounded function $f : [a, b] \to \mathbb{R}$. Here is one (non)example for Riemann integrability:

Example 31.4. Consider the function $f : [-2, 3] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 2|x| + 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

It is easy to show $f(x)$ is nowhere continuous by sequential argument. Moreover we know that $f(x) \geq 1$ for all $x \in \mathbb{Q}$. Therefore for any partition P of $[-2, 3]$ we can compute

$$L(f, P) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) = 0$$
$$U(f, P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) \geq \sum_{i=1}^{n} (x_i - x_{i-1}) = 3 - (-2) = 5$$

by the density of $\mathbb{Q}$ and $\mathbb{Q}^c$ in $\mathbb{R}$. Thus the function is not Riemann integrable on $[-2, 3]$ since we can choose $\epsilon_0 = 1$ then for any partition P of $[-2, 3]$ we always have $U(f, P) - L(f, P) \geq 5 > 1 = \epsilon_0$.

32 Quiz 10

1. (5 pts) Write down the 3rd order Taylor's formula, *i.e.* $n = 3$, centered at $a \in \mathbb{R}$ with Lagrange remainder **explicitly**, for a function $f \in C^3([a, b])$ and 4 times differentiable in (a, b).

   Proof. For any $x \in [a, b]$ we have

   $$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \frac{f^{(4)}(c)}{4!}(x - a)^4$$

   for some c between a and x. □

2. (5 pts) Let $f(x) = x^3 + x + 1$. **Apply the above result** to rewrite $f(x)$ as $f(x) = a_0 + a_1(x - 1) + a_2(x - 1)^2 + a_3(x - 1)^3$. In other words, express the function $f(x)$ in terms of powers of $(x - 1)$.

Proof. Since f is a polynomial function of degree 3 therefore $f^{(n)} = 0$ for all $n > 3$. Moreover $f'(x) = 3x^2 + 1$, $f''(x) = 6x$ and $f'''(x) = 6$ and thus $f(1) = 3, f'(1) = 4, f''(1) = 6$ and $f'''(1) = 6$. Use the Taylor's formula centered at $a = 1$ we get

$$f(x) = 3 + 4(x-1) + 3(x-1)^2 + (x-1)^3. \qquad \square$$

33 November 29

We have showed that

- A continuous function on $[a, b]$ is Riemann integrable.
- A monotonic function on $[a, b]$ is Riemann integrable.
- A bounded function with a single discontinuity is Riemann integrable since we can make the partitioned interval containing the discontinuity arbitrarily small.
- The proof of above can be easily generalized to show a bounded function with finitely many discontinuities is also Riemann integrable.
- For countably many discontinuities, the proof above cannot be applied, in this case one might need to use the compactness of $[a, b]$ to get the "finite covering" argument. But some special cases may not require the topological definition of compactness. For instance, if the set of discontinuities form a convergent sequence, then it still can be easily showed that the function f is Riemann integrable. We will see some examples later.
- Actually, we have the theorem due to Lebesgue, which says that $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if f is bounded and the set of points at which the function f is discontinuous has Lebesgue measure zero.

Example 33.1. Here are some (non)examples:

- Consider the floor function $f(x) = \lfloor x \rfloor$. Since f is monotonically increasing it is integrable on any $[a, b]$.
- Let $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0; \\ 1/n & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n} \text{ for some } n \in \mathbb{N}. \end{cases}$$

  Then again the function f is monotonically increasing on $[0, 1]$, thus it is Riemann integrable even if it has infinitely many discontinuities.
- Let $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}; \\ 1 & \text{otherwise.} \end{cases}$$

  Notice that the function is not continuous precisely at $\{1/n \mid n \in \mathbb{N}\} \cup \{0\}$. Now for any $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $1/N < \epsilon/2$. Now consider the intervals $[0, 1/N]$ and $[1/N, 1]$. Note that there are only finitely many $n \in \mathbb{N}$ such that $1 \leq n \leq N$ therefore f is integrable on $[1/N, 1]$. It means that there exists a partition P_ϵ of $[1/N, 1]$ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon/2.$$

Now consider the partition $P = P_\epsilon \cup \{0\}$, then

$$U(f, P) - L(f, P) = \frac{\epsilon}{2} - 0 + U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Thus we proved that f is integrable on $[0, 1]$. The proof can be generalized into any cases when the points of discontinuity form a convergent sequence.

- The Dirichlet function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$$

is not Riemann integrable. This can be easily shown using Riemann upper/lower sums.

- Consider a slightly modified function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1/2 & \text{if } x \text{ is rational;} \\ x & \text{if } x \text{ is irrational.} \end{cases}$$

For any partition P, we can assume P contains the mid point $1/2$ since adding this point will only make the upper sum smaller and lower sum greater (thus it doesn't affect the proof). Say

$$P = \{x_0 = 0, x_1, \cdots, x_n = 1/2, x_{n+1}, \cdots, x_{n+m}\}.$$

By the density of rationals and irrationals in $\mathbb{R}$, we have

(1) For all $i = 1, \cdots, n$,

$$M_i(f) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} = \frac{1}{2},$$
$$m_i(f) = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} = x_{i-1}.$$

(2) For all $j = 1, \cdots, m$,

$$M_{n+j}(f) = \sup\{f(x) \mid x \in [x_{n+j-1}, x_{n+j}]\} = x_{n+j},$$
$$m_{n+j}(f) = \inf\{f(x) \mid x \in [x_{n+j-1}, x_{n+j}]\} = \frac{1}{2}.$$

Therefore we can compute the upper and lower Riemann sums as follows:

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} \frac{1}{2}(x_i - x_{i-1}) + \sum_{j=1}^{m} x_{n+j}(x_{n+j} - x_{n+j-1}) \\ &\geq \frac{1}{2}\frac{1}{2} + \frac{1}{2}\sum_{j=1}^{m}(x_{n+j}^2 - x_{n+j-1}^2) \\ &= \frac{1}{4} + \frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{5}{8}, \end{aligned}$$

where the inequality is due to $x_{n+j} \geq \frac{x_{n+j}+x_{n+j-1}}{2}$. And

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}) + \sum_{j=1}^{m} \frac{1}{2}(x_{n+j} - x_{n+j-1}) \\ &\leq \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2) + \frac{1}{2}\frac{1}{2} \\ &= \frac{1}{2}\left(\frac{1}{4} - 0\right) + \frac{1}{4} = \frac{3}{8}, \end{aligned}$$

where the inequality is due to $x_{i-1} \leq \frac{x_i + x_{i-1}}{2}$. Thus we have

$$U(f,P) - L(f,P) \geq \frac{1}{4}$$

therefore the function f cannot be Riemann integrable on $[0,1]$.

34 December 1

We can also use upper/lower Riemann sums to find the integral of an integrable function. For example,

Example 34.1. Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x$. Let $P = \{x_0 = 0, x_1, \cdots, x_n = 1\}$ be a partition of $[0,1]$. It is easy to see that $M_i(f) = x_i$ and $m_i(f) = x_{i-1}$. Thus

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} x_i(x_i - x_{i-1}), \\ L(f,P) &= \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}). \end{aligned}$$

Then we get

$$U(f,P)+L(f,P)=\sum_{i=1}^{n}(x_i^2-x_{i-1}^2)=1,$$
$$U(f,P)-L(f,P)=\sum_{i=1}^{n}(x_i-x_{i-1})^2.$$

Therefore we can express the upper and lower Riemann sums separately as follows

$$U(f,P)=\frac{1}{2}+\frac{1}{2}\sum_{i=1}^{n}(x_i-x_{i-1})^2,$$
$$L(f,P)=\frac{1}{2}-\frac{1}{2}\sum_{i=1}^{n}(x_i-x_{i-1})^2.$$

In particular, we can partition $[0,1]$ evenly into n subintervals, denote by P_n such special partition. Then

$$\sum_{i=1}^{n}(x_i-x_{i-1})^2=\frac{n}{n^2}=\frac{1}{n}.$$

Thus

$$\frac{1}{2}-\frac{1}{2n}=L(f,P_n)\le\underline{\int_0^1} f\le\overline{\int_0^1} f\le U(f,P_n)=\frac{1}{2}+\frac{1}{2n}$$

and let $n\to\infty$ we obtain that f is integrable on $[0,1]$ and

$$\int_0^1 f(x)\mathrm{d}x=\frac{1}{2}.$$

Recall that the tagged partition (P,T) is a partition P of $[a,b]$ with tagged points in each subinterval at which the function is evaluated to contribute to the sum. We defined the Riemann sum $S(f,P,T)$ with respect to the tagged partition in the natural way without taking sup and inf of the function at each subinterval.

Now the function $f:[a,b]\to\mathbb{R}$ being Riemann integrable on $[a,b]$ is equivalent to saying there exists $I\in\mathbb{R}$ such that $\lim_{||P||\to 0}S(f,P,T)=I$ for any tag T of P, *i.e.* for any $\epsilon>0$ there exists $\delta>0$ such that for any partition P with $||P||<\delta$ we have $|S(f,P,T)-I|<\epsilon$ holds for any tag T of P.

The examples we have seen before using upper/lower Riemann sums to prove integrability can also be shown using Riemann sum.

Example 34.2. Consider the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1/n & \text{if } x=\frac{1}{n} \text{ for some } n\in\mathbb{N};\\ 0 & \text{otherwise.}\end{cases}$$

For any $\epsilon > 0$ define

$$F_\epsilon = \{x \in [0,1] \mid f(x) \geq \epsilon\}.$$

Then F_ϵ is a finite set and set $n_\epsilon = |F_\epsilon|$. Let (P,T) be any tagged partition of $[0,1]$ with $||P|| < \delta$ (we will determine δ in terms of ϵ later). Decompose (P,T) as (P_1,T_1) where the tags are in F_ϵ and (P_2,T_2) where the tags are in $[0,1] \setminus F_\epsilon$. Then

$$0 \leq S(f,P,T) = S(f,P_1,T_1) + S(f,P_2,T_2) \leq n_\epsilon\delta + \epsilon < 2\epsilon$$

if we set $n_\epsilon\delta < \epsilon$. This is because for $S(f,P_1,T_1)$ we have

$$S(f,P_1,T_1) \leq n_\epsilon \cdot 1 \cdot \delta$$

where n_ϵ meaning at most n_ϵ such points in T_1, 1 means $f(x) \leq 1$ and δ is the bound for the norm of P; For $S(f,P_2,T_2) < \epsilon$ it is because $f(x) < \epsilon$ by the assumption that T_2 are not chosen from F_ϵ. Therefore f is Riemann integrable on $[0,1]$ with $\int_0^1 f = 0$.

Using the similar idea we have prove the following fact:

Proposition 34.3. *Given $f : [a,b] \to \mathbb{R}$ bounded, and $f(x) = 0$ except for a finite number of points $c_1, \cdots, c_n \in [a,b]$, then f is Riemann integrable on $[a,b]$ and $\int_a^b f = 0$.*

Proof. Let $M = \max\{f(c_i) \mid i \in [n]\}$. For any $\epsilon > 0$ take $\delta = \frac{\epsilon}{Mn}$. Let (P,T) be any tagged partition with $||P|| < \delta$. Then we can decompose (P,T) into two disjoint parts: (P_1,T_1) as subpartition with tags in $\{c_i\}$; (P_2,T_2) as subpartition with tags not in $\{c_i\}$. Then

$$|S(f,P,T)| = |S(f,P_1,T_1) + S(f,P_2,T_2)| < n\frac{\epsilon}{nM}M + 0 = \epsilon.$$

Therefore f is Riemann integrable on $[a,b]$ with $\int_a^b f = 0$. $\square$

Remark 34.4. The above proposition can be rephrased as: if $f,g : [a,b] \to \mathbb{R}$ and $f(x) = g(x)$ except for a finite number of points in $[a,b]$ then f is integrable on $[a,b]$ if and only if g is integrable and in this case $\int_a^b f = \int_a^b g$. It means that changing a function at finite number of points does NOT affect the Riemann integrability and the value of integral.

BUT, the conclusion may fail if "finitely many points" is replaced by " countably infinite number of points". The followings are two simple reasons:

- We can turn a bounded function into an unbounded function by changing the values at countably infinite number of points. Consider $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} n & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}; \\ 0 & \text{otherwise.} \end{cases}$$

Such function f equals 0-function except at $1/n$. But f is unbounded so it cannot be Riemann integrable.

- The conclusion may still not hold if f is bounded. Take for example the Dirichlet function. The function is bounded and equals 0 except at $\mathbb{Q} \cap [0,1]$ which is countable. But the Dirichlet function is not Riemann integrable.

35 Quiz 11

1. (5 pts) Let $a, b \in \mathbb{R}$ with $a < b$, and let $P = \{x_0 = a, x_1, \cdots, x_{n-1}, x_n = b\}$ be a partition of the interval $[a, b]$. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded.
   a) (3 pts) State the definition of *upper Riemann sum* of f with respect to the partition P explicitly.

   Proof. The upper Riemann sum of f with respect to the partition P is defined to be
   $$U(f, P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$
   where $M_i(f) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$. □

   b) (2 pts) State explicitly the definition of *upper integral* of f on $[a, b]$.

   Proof. The upper integral of f on $[a, b]$ is
   $$\overline{\int_a^b} f = \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}.$$ □

2. (5 pts) Suppose that $f(x) = 0$ if $x \in [0, 1)$, $f(x) = 1$ if $x \in [1, 2)$, and $f(x) = 2$ if $x \in [2, 3]$. Give an example of a partition P of the interval $[0, 3]$ for which $U(f, P) - L(f, P) < 0.1$.

Proof. The function $f(x)$ has two jump discontinuities, one at $x = 1$ the other at $x = 2$. Therefore the nonzero difference of $U(f, P)$ and $L(f, P)$ only happens at those subintervals (given by the partition) containing $x = 1, 2$ as interior points, or including these two points as the right endpoints. Start with the following partition,
$$P = \{x_0 = 0, x_1, x_2 = 1, x_3, x_4 = 2, x_5 = 3\}$$
and assume for simplicity $\Delta x := x_2 - x_1 = x_4 - x_3$. Therefore the difference is given by
$$U(f, P) - L(f, P) = (1 - 0)\Delta x + (2 - 1)\Delta x = 2\Delta x$$
since on other subintervals, namely on $[0, x_1], [x_2, x_3], [2, 3]$ the sup and inf of $f(x)$ are the same. So in order to make $U(f, P) - L(f, P) < 1/10$, we only need to make $2\Delta x < 1/10$, i.e. $\Delta x < 1/20$. Therefore we can take for example
$$P = \{x_0 = 0, x_1 = 0.99, x_2 = 1, x_3 = 1.99, x_4 = 2, x_5 = 3\}.$$ □

36 December 6

Recall the fundamental theorem of calculus:

- (FTC I) Let f be a Riemann integrable function on $[a, b]$. Define F on $[a, b]$ by

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

Then $F(x)$ is continuous on $[a, b]$. Moreover if f is continuous at a point $c \in [a, b]$ then F is differentiable at c and $F'(c) = f(c)$.

Remark 36.1. (1) Function F is uniquely determined up to addition of a constant.

(2) FTC I tells us that any continuous function has an antiderivative which is differentiable. But sometimes it is not easy to write down the formula explicitly, for example $f(x) = e^{-x^2}$.
A discontinuous function may have an antiderivative. Consider

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Note that this function $F(x)$ is differentiable but not continuously differentiable. Let $f(x) = F'(x)$, *i.e.* F is an antiderivative of f. Note that f is not continuous at zero because F' is not. So it is possible for a function f which has discontinuity at some points but still admits an antiderivative. BUT, by Darboux theorem (IVP for derivative), if f is the derivative of F on some interval, then f has intermediate value property (no matter f is continuous or not). This tells us that a discontinuous function without IVP does NOT have an antiderivative. For example, a function with jump discontinuities such as floor function.

Example 36.2. Let $f : [0, 2] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1; \\ 1 & \text{if } x = 1. \end{cases}$$

Does there exist a function $F : [0, 2] \to \mathbb{R}$ such that $F' = f$ on $[0, 2]$?
The answer is no. Assume such F exists. Since $f(x) = 0$ on $(0, 1)$ then $F = a$ constant in $(0, 1)$; likewise $F = b$ constant in $(1, 2)$. But if F exists it has to be continuous on $[0, 2]$, in particular continuous at 1. It forces $a = b = f(1) = 1$. Thus F is constant in $(0, 2)$. It leads to a contradiction $F'(1) = f(1) = 1 = 0$. Therefore such function does not exist.

(3) The differentiability of F at c only depends on the continuity of f at the same point c. But the continuity of F' at c doesn't come from

differentiablity of f at c. To be precise, let's assume $f : [a, b] \to \mathbb{R}$ Riemann integrable and define $F(x)$ on $[a, b]$ as in FTC I. Let $c \in (a, b)$ be an interior point.

Example 36.3. If f is differentiable at c, then F is differentiable at c but may not be continuously differentiable, *i.e.* F' may not be continuous at c. Because we only have $F'(c) = f(c)$ holds at point c, not in some neighborhood of c but continuity is a local property. The following example shows that F' may not exist in any arbitrarily small neighborhood of c, *i.e.* F is not differentiable in any small neighborhood of c.
Define

$$f(x) = \begin{cases} 1 & \text{if } x \in (-\infty, -1] \cup [1, \infty); \\ \frac{1}{2^2} & \text{if } x \in (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1); \\ \frac{1}{3^2} & \text{if } x \in (-\frac{1}{2}, -\frac{1}{3}] \cup [\frac{1}{3}, \frac{1}{2}); \\ \vdots & \\ \frac{1}{n^2} & \text{if } x \in (-\frac{1}{n-1}, -\frac{1}{n}] \cup [\frac{1}{n}, \frac{1}{n-1}); \\ \vdots & \\ 0 & \text{if } x = 0. \end{cases}$$

It is easy to show by definition that f is differentiable at 0 with $f'(0) = 0$ (Exercise). Due to symmetry we only need to consider $f(x)$ on $[0, \infty)$, the other half is similar. Since f is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$ it is Riemann integrable on $\mathbb{R}$. And $F(x) = \int_0^x f(t) \mathrm{d}r$ is piecewise linear with different slopes given by the value of f. So F has sharp corners precisely at $\{\pm 1/n\}$. It can be checked that F' does not exist at those $\{\pm 1/n\}$. And thus F' cannot be continuous at 0 by using sequential criterion of continuity and taking sequence $x_n = 1/n$ for example.

(4) FTC I also provides a way to define transcendental functions *i.e.* functions not expressed as a finite combination of algebraic operations, as integral of elementary functions.
For example, consider

$$\ln(x) = \int_1^x \frac{1}{t} \mathrm{d}t$$

where the integral is well-defined on $(0, \infty)$ since $1/t$ is continuous on $[1, x]$ for $x > 1$ or $[x, 1]$ for $0 < x < 1$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$$

and we can deduce the properties of $\ln(x)$ from this presentation. For instance,

$$\ln(x)+\ln(y)=\int_1^x \frac{\mathrm{d}t}{t}+\int_1^y \frac{\mathrm{d}t}{t}=\int_1^x \frac{\mathrm{d}t}{t}+\int_x^{xy} \frac{\mathrm{d}s}{s}=\int_1^{xy} \frac{\mathrm{d}t}{t}=\ln(xy)$$

where we are using substitution $s=xt$.

- (FTC II) Suppose f is continuous on $[a,b]$ and differentiable in (a,b) with f' being Riemann integrable on $[a,b]$. Then

$$\int_a^b f'(x)\mathrm{d}x=f(b)-f(a).$$

Remark 36.4. (1) It is not necessary to assume the existence of the right derivative of f at b or the left derivative of f at b.

(2) FTC II gives a way to find the exact value of an integral as long as we can find an antiderivative of the integrand.

(3) One example of using FTC II to find the value of infinite sum. For example we consider $f:[1,2]\to\mathbb{R}$ and $f(x)=1/x$. Then f is integrable because it is continuous. Let $n\in\mathbb{N}$ take partition P_n with tags T_n by taking left endpoints:

$$\begin{aligned}P_n&=\{1,1+1/n,\cdots,1+k/n,\cdots,1+n/n\},\\ T_n&=\{1+k/n \mid k=0,\cdots,n-1\}.\end{aligned}$$

Then $||P_n||\to 0$ as $n\to\infty$. The Riemann sum can be computed as

$$S(f,P_n,T_n)=\sum_{k=1}^n \frac{1}{1+\frac{k-1}{n}}\frac{1}{n}=\sum_{k=1}^n \frac{1}{n+k-1}.$$

Thus

$$\lim_{n\to\infty}\sum_{k=1}^n \frac{1}{n+k-1}\to\int_1^2 \frac{1}{x}\mathrm{d}x=\ln(2)$$

by FTC II.

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1. With an example show that the identity $\overline{\int}_0^1 f+\overline{\int}_0^1 g=\overline{\int}_0^1 (f+g)$ may be false if f or g are not integrable.

Proof. Let $f,g:[0,1]\to\mathbb{R}$ with f being the Dirichlet function and $g=1-f$:

$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases},\quad g(x)=\begin{cases}0 & \text{if } x\in\mathbb{Q}\\ 1 & \text{if } x\notin\mathbb{Q}\end{cases}.$$

Note that both f and g are not Riemann integrable on $[0,1]$. And $f+g=1$ so

$$\overline{\int_0^1} f = \overline{\int_0^1} g = 1, \quad \overline{\int_0^1} (f+g) = 1.$$

Thus

$$\overline{\int_0^1} f + \overline{\int_0^1} g = 2 > \overline{\int_0^1} (f+g) = 1.$$ □

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2|x|+1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Show that f is not Riemann integrable on $[-1,2]$.

Proof. First note that $f(x) \geq 1$ for all $x \in \mathbb{Q}$. By the density of $\mathbb{Q}$ and $\mathbb{Q}^c$ in $\mathbb{R}$, for any partition $P = \{x_0 = -1, x_1, \cdots, x_n = 2\}$ of $[-1,2]$, we can compute its corresponding upper/lower Riemann sums:

$$\begin{aligned} U(f,P) &\geq \sum_{i=1}^{n} (x_i - x_{i-1}) = 2 - (-1) = 3; \\ L(f,P) &= 0. \end{aligned}$$

Therefore $U(f,P) - L(f,P) \geq 3$ for any partition P of $[-1,2]$ which implies f is not Riemann integrable on $[-1,2]$. □

3. Consider the function $f(x) = \lfloor x \rfloor$. Using theorems we proved in class explain why $f \in \mathcal{R}([0,4])$. For each $\epsilon > 0$ find an explicit n so that a standard partition $\mathcal{P}_n$ of $[0,4]$ satisfies $U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) < \frac{1}{100}$.

Proof. The function is increasing on $[0,4]$ thus it is Riemann integrable on $[0,4]$. Or using the fact that such function only has finitely many discontinuities on $[0,4]$. To make the inequality hold for $\epsilon = 1/100$, take for example $n = 4000$:

$$\mathcal{P}_{4000} = \left\{0, \frac{1}{1000}, \cdots, \frac{999}{1000}, 1, \frac{1001}{1000}, \cdots, \frac{1999}{1000}, 2, \frac{2001}{1000}, \cdots, \frac{2999}{1000}, 3, \cdots, 4\right\}.$$

Then

$$U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) = \frac{4}{1000} < \frac{1}{100}.$$ □

4. We mentioned in class that Thomae's function

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ with } p,q \in \mathbb{N} \text{ coprime} \\ 0 & \text{otherwise} \end{cases}$$

is Riemann integrable on $[0,1]$. Show that $\int_0^1 f = 0$.

Proof. By density of irrationals, we know $L(f, P) = 0$ for any partition of $[0, 1]$. For any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $1/N < \epsilon/2$. Consider the set

$$F_\epsilon = \{x \in [0, 1] \mid f(x) \geq \epsilon/2\}.$$

Then the set F_ϵ is finite since there are only finitely many rational numbers in $[0, 1]$ with denominator less and equal to N. Let $|F_\epsilon| = l$. Now take $P_\epsilon = \{x_0 = 0, x_1, \cdots, x_n = 1\}$ be a partition of $[0, 1]$ such that $||P_\epsilon|| < \epsilon/4l$. Then

$$U(f, P_\epsilon) < 1 \cdot 2l \cdot \frac{\epsilon}{4l} + \frac{\epsilon}{2} \cdot (1 - 0) = \epsilon.$$

Therefore the Thomae's function is Riemann integrable on $[0, 1]$. Since the lower Riemann sum is always 0 then the lower integral is 0, and the function is proved to be Riemann integrable therefore $\int_0^1 f = 0$. □

5. Prove that if $f : [a, b] \to \mathbb{R}$, where $a < b$, is continuous and nonnegative on $[a, b]$ then

$$\int_a^b f = 0 \implies f = 0 \text{ on } [a, b].$$

Proof. By way of contradiction suppose $f(c) > 0$ for some $c \in [a, b]$. I will assume $c \in (a, b)$ the endpoint case is similar. By assumption f is continuous so there exists $\delta > 0$ sufficiently small such that $(c - \delta, c + \delta) \subseteq (a, b)$ and

$$|f(x) - f(c)| < \frac{f(c)}{2}$$

whenever $x \in (c - \delta, c + \delta)$. Then

$$f(x) = f(c) + f(x) - f(c) \geq f(c) - |f(x) - f(c)| \geq f(c) - \frac{f(c)}{2} = \frac{f(c)}{2}$$

for $x \in (c - \delta, c + \delta)$. Therefore

$$\int_a^b f = \int_a^{c-\delta} f + \int_{c-\delta}^{c+\delta} f + \int_{c+\delta}^b f \geq 2\delta \frac{f(c)}{2} > 0$$

which is a contradiction. Thus such c does not exist. □