COMPREHENSIVE EXAM

ZHENGNING HU

1 QUESTIONS FROM DAN EDIDIN

From the paper *The Chow ring of the stack of hyperelliptic curves of odd genus* by Andrea Di Lorenzo.

1. What is a GL_3-counterpart of a scheme Y with a PGL_2 action? Prove that there is an embedding of PGL_2 into GL_3 to conclude that $\mathrm{GL}_3 / \mathrm{PGL}_2$ is a GL_3-counterpart of $\operatorname{Spec} k$. (Hint: think about the adjoint representation.)

   ***Answer*:** Note that when we say a scheme, it is always of finite type over $\operatorname{Spec} k$. Given a scheme Y with a PGL_2 action, a GL_3-counterpart of Y is a scheme X endowed with a GL_3 action such that $[X/\mathrm{GL}_3] \cong [Y/\mathrm{PGL}_2]$ as quotient stacks, i.e., over any k-scheme S, $[X/\mathrm{GL}_3](S) \cong [Y/\mathrm{PGL}_2](S)$ is an equivalent of categories. A quotient stack $[X/G]$ is a stack of G-torsors with an equivariant morphism on X. No matter whether the group scheme G acts freely on X or not, $X \to [X/G]$ always forms a G-torsor. One of the simplest examples for quotient stack is $[\operatorname{Spec} k/G]$ which is just a stack of G-torsors and it is the classifying stack $\mathcal{B}G$.

   The idea of constructing the GL_3-counterpart of a PGL_2-scheme is that given a morphism of group schemes $H \to G$, it induces a functor F from the category of H-schemes to the category of G-schemes such that $[X/H] \cong [F(X)/G]$ is an isomorphism.

   Now there are two questions we can ask: Is it legitimate to define a GL_3-counterpart of a PGL_2-scheme? Or in other words, does a GL_3-counterpart always exist? If it is, what is the main advantage of taking the GL_3-counterpart?

   Suppose that we have a morphism of group schemes $\rho : H \to G$ and a scheme X with an action of H, then we get a H-torsor $f : X \to [X/H]$, which is, étale locally isomorphic to $U \times H$. Intuitively, we can view the torsor f as defined by a covering $\{U_i\}$ with a collection of transition functions $\{\phi_{ij}\}$ encoding the gluing data. Then the covering $\{U_i\}$ with G-valued functions $\{\rho \circ \phi_{ij}\}$ will define a G-torsor $X \times^H G \to [X/H]$ over the quotient stack $[X/H]$ where the G-scheme $X \times^H G = (X \times G)/H$ endowed with a right diagonal action of H is indeed a G-counterpart of X.

   Consider the terminal object $\operatorname{Spec} k$ in the category **Sch**, equipped with the trivial action of PGL_2. If one can find a GL_3-counterpart of $\operatorname{Spec} k$, denoted by $\mathcal{S}$, then by pulling back along the GL_3-torsor $\mathcal{S} \to \mathcal{B}\,\mathrm{PGL}_2$ of the scheme $X \to \operatorname{Spec} k$ representing some stack, one can easily get its GL_3-counterpart. Therefore it boils down to finding a GL_3-counterpart of $\operatorname{Spec} k$. First of all since PGL_2 and SL_2 share the same Lie algebra, a 3-dimensional vector space consisting of traceless 2×2 matrices with basis

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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We can define the adjoint representation of PGL_2,

$$\begin{aligned}\rho : \mathrm{PGL}_2 &\longrightarrow \mathrm{GL}(\mathfrak{pgl}_2) = \mathrm{GL}_3 \\ [A] &\longmapsto (X \mapsto AXA^{-1}).\end{aligned}$$

It is indeed a morphism of algebraic group schemes since it is defined by polynomials. As mentioned before, we view $\operatorname{Spec} k$ as a PGL_2-scheme with trivial action. Then we get a PGL_2-torsor $\operatorname{Spec} k \to [\operatorname{Spec} k/\mathrm{PGL}_2]$ over the classifying stack $\mathcal{B}\,\mathrm{PGL}_2$. Together with the above adjoint representation ρ of PGL_2, the induced GL_3-torsor is given by

$$(\operatorname{Spec} k \times_{\operatorname{Spec} k} \mathrm{GL}_3)/PGL_2 \cong \mathrm{GL}_3 / \mathrm{PGL}_2 \longrightarrow \mathcal{B}\,\mathrm{PGL}_2$$

where the GL_3-counterpart is the right quotient space $\operatorname{Spec} k \times^{\mathrm{PGL}_2} \mathrm{GL}_3$ with (right) action under $(x, g) \sim (x, g\rho(h)) = (x, \rho(h)^{-1}g)$ for $g \in \mathrm{GL}_3$, $h \in \mathrm{PGL}_2$. Therefore we proved $\mathrm{GL}_3 / \mathrm{PGL}_2$ is a GL_3-counterpart of $\operatorname{Spec} k$.

One of the advantages to finding a GL_3-counterpart of some PGL_2-scheme is that PGL_2 is not a special group where some equivariant computation may be hard to carry on. It means that there is some PGL_2-torsor that is not Zariski-locally trivial but only étale-locally trivial. For instance, since the construction of GL_3-counterpart is functorial, we can pass the computation of the equivariant Chow ring $A_*^{\mathrm{PGL}_2}(X)$ to the computation of $A_*^{\mathrm{GL}_3}(Y)$ where Y is a GL_3-counterpart of X. In this paper, the author is using a GL_3-counterpart of $\mathcal{H}_g$ when g is odd, to compute its integral Chow ring.

2. Let $\mathcal{S}$ be the set of forms of degree 2 in 3 variables which are smooth. Prove that $\mathcal{S}$ is a GL_3-counterpart of $\operatorname{Spec} k$. Does this automatically imply that $\mathcal{S}$ is isomorphic to $\mathrm{GL}_3 / \mathrm{PGL}_2$?

***Answer*:** We denote by $\mathcal{S}$ the set of smooth ternary quadratic forms. It is an open subscheme of the affine space $\mathbb{A}(2,2)$, the space of ternary quadratic forms. Before proving $\mathcal{S}$ is a GL_3-counterpart of $\operatorname{Spec} k$, first recall that $\mathcal{M}_0$ is the moduli stack of smooth rational curves. Since all rational curves are isomorphic, $\mathcal{M}_0$ is just a point as a moduli space, but it consists of more information as a stack and moreover,

$$\mathcal{M}_0 \cong \mathcal{B}\,\mathrm{Aut}(\mathbb{P}^1) \cong \mathcal{B}\,\mathrm{PGL}_2 \cong [\operatorname{Spec} k/\mathrm{PGL}_2].$$

It suffices to prove that the scheme $\mathcal{S}$ with action of GL_3 given by

$$A \cdot q(\underline{x}) = (\det A) q(A^{-1}\underline{x}), \quad \underline{x} = (x_0, x_1, x_2),$$

for $A \in \mathrm{GL}_3$ and $q(\underline{x}) \in \mathcal{S}$, is a GL_3-torsor of $\mathcal{M}_0$. The idea is to construct prestacks in groupoids over **Sch** which are easy to see being GL_3-torsors over $\mathcal{M}_0$ and also being isomorphic to $\mathcal{S}$.

There is a general result by cohomology and base change which is commonly used in moduli theory. Given a family of smooth curves of genus g, namely $\pi : \mathcal{C} \to S$, which is a flat and proper morphism, the pushforward $\pi_*\omega_{\mathcal{C}/S}^{\otimes k}$ when $k \geq 1$ is locally free of rank

$$\begin{aligned}h^0(\mathcal{C}_s, \omega_{\mathcal{C}_s}^{\otimes k}) &= \deg(\omega_{\mathcal{C}_s}^{\otimes k}) - g + 1 \\ &= (2k-1)(g-1)\end{aligned}$$

where $h^1(\mathcal{C}_s, \omega_{\mathcal{C}_s}^{1-k}) = 0$. Thus it yields an embedding $\mathcal{C} \to \mathbb{P}(\pi_*\omega_{\mathcal{C}/S}^{\otimes k})$.

Given a family of smooth rational curves $\pi : \mathcal{C} \to S$, take $\pi_*\omega_{\mathcal{C}/S}^{\vee}$, which is a locally free sheaf of rank 3 since by checking the geometric point $s \in S$, we have $h^1(\mathcal{C}_s, \omega_{\mathcal{C}_s}^{\vee}) = 0$ by Serre duality and thus $h^0(\mathcal{C}_s, \omega_{\mathcal{C}_s}^{\vee}) = 2 + 1 = 3$. We then obtain an embedding

$$i : \mathcal{C} \hookrightarrow \mathbb{P}(\pi_*\omega_{\mathcal{C}/S}^{\vee}) = \mathbb{P}^2_S.$$

Over any geometric point of the family S, it is determined by the complete linear system $|\mathcal{O}(2)|$.

By using the above fact about the rank 3 vector bundle over scheme S, we can construct the following GL_3-torsor $\mathcal{E}$ over $\mathcal{M}_0$. Over a scheme $S \in \mathbf{Sch}$, it is a groupoid

$$\mathcal{E}(S) = \{\pi : \mathcal{C} \to S, \alpha : \pi_* \omega_\pi^\vee \cong \mathcal{O}_S^{\oplus 3}\}$$

with a natural GL_3 action on the rank 3 bundles α. Thus $\mathcal{E}$ forms a GL_3-torsor over $\mathcal{M}_0$. But it is still hard to see $\mathcal{E} \cong \mathcal{S}$. Now we define another intermediate prestack $\mathcal{E}'$. Over a scheme S, it is defined by

$$\mathcal{E}'(S) = \left\{ \left(\begin{array}{ccc} \mathcal{C} & \xhookrightarrow{i} & \mathbb{P}^2_S \\ \downarrow \pi & \swarrow & \\ S & & \end{array}, \beta : i^* \mathcal{O}_{\mathbb{P}^2_S}(1) \cong \omega_\pi^\vee \right) \right\}$$

and it can also be written as

$$\mathcal{E}'(S) = \{\pi : \mathcal{C} \to S, \pi_* \omega_\pi^\vee \cong H^0(\mathbb{P}^2_S, \mathcal{O}(1)) \otimes \mathcal{O}_S\}.$$

And $\mathcal{E} \cong \mathcal{E}'$. Therefore it suffices to check $\mathcal{E}' \cong \mathcal{S}$. Giving a morphism a sheaves is equivalent to giving a global section, so the isomorphism of sheaves β can be viewed as a nonzero section of $H^0(\mathcal{C}, i^*\mathcal{O}(1) \otimes \omega_\pi)$. Let $\mathcal{I}$ be the ideal sheaf of the image $i(\mathcal{C})$ in $\mathbb{P}^2_S$. Then

$$\begin{aligned} H^0(\mathcal{C}, i^*\mathcal{O}(1) \otimes \omega_\pi) &= H^0(\mathbb{P}^2_S, i_*(i^*\mathcal{O}(1) \otimes \omega_\pi)) \\ &= H^0(\mathbb{P}^2_S, i_*(i^*(\mathcal{O}(1) \otimes \omega_{\mathbb{P}^2_S/S} \otimes \mathcal{I}^\vee))) \\ &= H^0(\mathbb{P}^2_S, \mathcal{O}(-2) \otimes \mathcal{I}^\vee \otimes i_*\mathcal{O}_\mathcal{C}). \end{aligned}$$

Consider the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^2_S} \to i_*\mathcal{O}_\mathcal{C} \to 0.$$

Twist by $\mathcal{O}(-2) \otimes \mathcal{I}^\vee$ and take the long exact sequence of cohomology, we get

$$H^0(\mathbb{P}^2_S, \mathcal{O}(-2) \otimes \mathcal{I}^\vee) = H^0(\mathbb{P}^2_S, i_*\mathcal{O}_\mathcal{C} \otimes \mathcal{O}(-2) \otimes \mathcal{I}^\vee).$$

It means that a nonzero global section of $i^*\mathcal{O}_\mathcal{C} \otimes \mathcal{O}(-2) \otimes \mathcal{I}^\vee$ induces a section of $\mathcal{O}(-2) \otimes \mathcal{I}^\vee$ which induces an isomorphism $\mathcal{O}(-2) \cong \mathcal{I}$ canonically. Since we have an injective morphism $\mathcal{O}(-2) \hookrightarrow \mathcal{O}_{\mathbb{P}^2_S}$, which yields a morphism of sheaves $\mathcal{O} \to \mathcal{O}(2)$ and it determines canonically a global section q in $H^0(\mathbb{P}^2_S, \mathcal{O}(2))$. The canonically defined section q has smooth zero locus in $\mathbb{P}^2_S$ and it can viewed as an element in $\mathcal{S}$. Conversely, given a section $q \in H^0(\mathbb{P}^2_S, \mathcal{O}(2))$, or equivalently, given a smooth ternary quadratic form, with zero locus $Q \subset \mathbb{P}^2_S$, since $\mathcal{I}_Q \cong \mathcal{O}(-2) \cong \omega_{\mathbb{P}^2_S}(1)$ we can construct the isomorphism β via

$$\omega_\pi^\vee = i^*(\omega_{\mathbb{P}^2_S}^\vee \otimes \mathcal{I}) \cong i^*\mathcal{O}(1).$$

The action of GL_3 on $\mathcal{S}$ is compatible with that on the invertible sheaves $\mathcal{I}_Q \cong \omega_{\mathbb{P}^2_S}(1)$ induced by the corresponding action on $\mathbb{P}^2_S = \mathbb{P}(V_S)$ where V_S is a three dimensional vector space over S, since $\omega_{\mathbb{P}^2_S}(1) \cong \mathcal{O}(-2) \otimes \det V_S$.

Now we have shown that $\mathcal{S}$ with the GL_3 action defined as above is indeed a GL_3-counterpart of $\operatorname{Spec} k$. Together with the result from **Question 1**, we get two GL_3-torsors over $\mathcal{M}_0$. It does NOT automatically imply that $\mathrm{GL}_3 / \mathrm{PGL}_2$ is isomorphic to $\mathcal{S}$ as k-schemes. But if we can prove one of the followings holds, then we can deduce this isomorphism:

(1) There is a GL_3-equivariant morphism $\mathrm{GL}_3 / \mathrm{PGL}_2 \to \mathcal{S}$;

(2) The only faithful 3-dimensional representation of PGL_2 is the adjoint representation. Or alternatively, since the representation of the quotient is deduced from the original group, it suffices to check that the only 3-dimensional representation of GL_2 which acts trivially on its center is the adjoint representation. Note that this approach heavily relies on the ground field k.

Let's first look at the second statement (2). One fact from representation theory about the representation of GL_2 over an infinite field says that all irreducible representations of GL_2 have the form E^λ as a Schur module. Moreover since PGL_2 is the quotient of GL_2 modulo the nonzero scalar matrices, the determinant representation of PGL_2 must be trivial. Therefore the only 3-dimensional representation of GL_2 acting trivially on its center must be the (irreducible) adjoint representation associated to the Schur module $E^{(2,2)}$ with basis as semi-standard tableaux of shape $(2,2)$ with entries chosen from $\{1,2,3\}$.

Let's then turn to the first statement. It is easy to write the embedding $\rho : \mathrm{PGL}_2 \hookrightarrow \mathrm{GL}_3$ explicitly with the basis e_1, e_2, e_3 defined in **Question 1**. Namely,

$$\rho : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \frac{1}{ad-bc} \begin{pmatrix} a^2 & -b^2 & -2ab \\ -c^2 & d^2 & 2cd \\ -ac & bd & ad+bc \end{pmatrix}.$$

Now defined a PGL_2-invariant ternary quadratic form $f \in \mathcal{S} = \mathbb{A}_{\mathrm{sm}}(2,2)$ by

$$f(\underline{x}) = x_0x_1 + x_2^2, \quad \underline{x} = (x_0, x_1, x_2).$$

Then it can be checked that $\mathrm{PGL}_2 = \mathrm{GL}_2 / \mathbb{G}_m$ is isomorphic as groups to

$$O(f) = \{T \in \mathrm{GL}_3 \mid f(T\underline{x}) = f(\underline{x}), \det T = 1\}.$$

Thus we can define the map

$$\begin{aligned} \mathrm{GL}_3 &\longrightarrow \mathcal{S} \\ T &\longmapsto f(T\underline{x}) \end{aligned}$$

which is of course a morphism of (quasi)-affine varieties and with elements in $\mathrm{PGL}_2 \subset \mathrm{GL}_3$ fixing $f(\underline{x})$. So this map factors through the quotient $\mathrm{GL}_3 / \mathrm{PGL}_2 \to \mathcal{S}$. One should also check details such as that the discriminant locus of $\mathbb{A}(2,2)$ is determined fully by the nonvanishing of determinants of elements in GL_3, and the morphism is GL_3-equivariant, etc. So that we obtain a GL_3-equivariant well-defined moprhism $\mathrm{GL}_3 / \mathrm{PGL}_2 \to \mathcal{S}$ where both are GL_3-torsors over $\mathcal{M}_0 \cong \mathcal{B}\,\mathrm{PGL}_2$ and then they have to be isomorphic by descent. To be precise, we could base change to a suitable étale cover U, and thus the GL_3-equivariant morphism becomes $U \times \mathrm{GL}_3 \to U \times \mathrm{GL}_3$ between two trivial GL_3-bundles defined by multiplication to the second factor by some element in GL_3. So it is an isomorphism and descends to the original equivariant map to be an isomorphism as well.

3. Let $\mathbb{A}(1,2n)$ be the vector space of homogeneous polynomials of degree $2n$ with the action of GL_2 given by

$$A \cdot f(x,y) = \det(A)^n f(A^{-1}(x,y)).$$

Prove that this action descends to an action of PGL_2 and describe with some details the GL_3-counterpart of this space as well as the GL_3-counterpart of $\mathbb{P}(\mathbb{A}(1,2n))$.

***Answer*:** First of all let us prove the action of GL_2 descends to an action of PGL_2. Two elements $A, A' \in \mathrm{GL}_2$ are equivalent in PGL_2 if and only if there exists some nonzero scalar matrix λI with $\lambda \in \mathbb{G}_m$ such that $A' = \lambda A$. Define the action of PGL_2 on $\mathbb{A}(1,2n)$ by

$$[A] \cdot f(x,y) = \det(A)^n f(A^{-1}(x,y)).$$

In order to prove the GL_2-action descends to the action of PGL_2 above, we only need to check $[A]\cdot f(x,y) = [A']\cdot f(x,y)$. This is easy to check since

$$\begin{aligned}[A']\cdot f(x,y) &= \det(A')^n f((A')^{-1}(x,y))\\ &= (\lambda^2)^n \det(A)^n f(\lambda^{-1}A^{-1}(x,y))\\ &= \lambda^{2n}\det(A)^n \lambda^{-2n} f(A^{-1}(x,y))\\ &= \det(A)^n f(A^{-1}(x,y))\\ &= [A]\cdot f(x,y).\end{aligned}$$

Next we want to find the GL_3-counterpart of $\mathbb{A}(1,2n)$ as well as its projectivization $\mathbb{P}(\mathbb{A}(1,2n))$.

By the functorial construction of GL_3-counterpart, our goal is to find a nice and explicit description of the top-left corner of the following Cartesian diagram

$$\begin{array}{ccc} [\mathbb{A}(1,2n)/\operatorname{PGL}_2]\times_{\mathcal{M}_0}\mathcal{S} & \longrightarrow & \mathcal{S}=\mathbb{A}_{\mathrm{sm}}(2,2)\\ \downarrow & & \downarrow\\ [\mathbb{A}(1,2n)/\operatorname{PGL}_2] & \longrightarrow & \mathcal{M}_0=[\operatorname{Spec}k/\operatorname{PGL}_2]\end{array}$$

where $[\mathbb{A}(1,2n)/\operatorname{PGL}_2]$ is the quotient stack and the action of PGL_2 on $\mathbb{A}(1,2n)$ is induced from the action of GL_2 as we argue in the first part of this question. Recall that $\mathbb{A}(n,m)$ is the scheme representing the sheaf sending a scheme S to the space of global sections $H^0(\mathbb{P}^n_S,\mathcal{O}(m))$. Over any k-scheme S, a degree $2n$ binary form in $\mathbb{A}(1,2n)(S)$ can be viewed as a global section in $H^0(\mathbb{P}^1_S,\mathcal{O}(2n))$, then it allows us to consider a rigidified auxiliary stack of $\mathbb{A}(1,2n)$:

$$\widetilde{\mathbb{A}(1,2n)}(S) := \{\pi:\mathcal{C}\to S, \phi:\mathcal{C}\cong\mathbb{P}^1_S, \sigma\in H^0(\mathcal{C},\omega_\pi^{-\otimes n})\}.$$

The free actions of PGL_2 are compatible since $\omega_\pi^{-\otimes n}\cong\mathcal{O}(2n)\otimes(\det V_S)^n$ for $\mathbb{P}^1_S=\mathbb{P}(V_S)$. Moreover

$$[\widetilde{\mathbb{A}(1,2n)}/\operatorname{PGL}_2](S) = \{\pi:\mathcal{C}\to S,\sigma\}.$$

Now it reduces to considering the fiber product $[\widetilde{\mathbb{A}(1,2n)}/\operatorname{PGL}_2]\times_{\mathcal{M}_0}\mathcal{E}'$. Its objects over a scheme S are

$$\left\{\left(\begin{array}{ccc}\mathcal{C} & \overset{i}{\hookrightarrow} & \mathbb{P}^2_S\\ \downarrow{\scriptstyle\pi} & \swarrow & \\ S & & \end{array}, \sigma\in H^0(\mathcal{C},\mathcal{T}^n_{\mathcal{C}/S}), \beta: i^*\mathcal{O}(1)\cong\mathcal{T}_{\mathcal{C}/S}\right)\right\}.$$

Note that given a family of rational curves $\pi:\mathcal{C}\to S$, a section $q\in H^0(\mathbb{P}^2_S,\mathcal{O}(2))$, or an element in $\mathcal{S}$ is equivalent to a pair (i,β). And moreover, under the isomorphism $\beta: i^*\mathcal{O}(1)\cong\mathcal{T}_{\mathcal{C}/S}$, a global section $\sigma\in H^0(\mathcal{C},\mathcal{T}^n_{\mathcal{C}/S})$ is given by $f\in H^0(\mathcal{C},i^*\mathcal{O}(n)\cong\mathcal{O}_{\mathcal{C}}(n))$. Combine all these facts, we can define the GL_3-torsor over $[\mathbb{A}(1,2n)/\operatorname{PGL}_2]$ as the scheme V_n representing a stack in sets over **Sch** by

$$V_n(S) = \{(q,f)\}$$

where $q\in H^0(\mathbb{P}^2_S,\mathcal{O}(2))$ with zero locus $Q\subset\mathbb{P}^2_S$ being smooth over S, and $f\in H^0(Q,\mathcal{O}_Q(n))$. Therefore such V_n is a GL_3-counterpart of $\mathbb{A}(1,2n)$ with action of GL_3 defined by

$$A\cdot(q(\underline{x}),f(\underline{x})) = (\det(A)q(A^{-1}\underline{x}), f(A^{-1}\underline{x})).$$

Likewise we also have that V_n is a $\mathrm{GL}_3\times\mathbb{G}_m$-counterpart of $\mathbb{A}(1,2n)$ by considering the action of $\mathbb{G}_m$ on V_n as multiplication on the second factor and on $\mathbb{A}(1,2n)$ by multiplication.

Note that the vector bundle V_n defined as above can also be viewed as the sheafification of the sheaf defined by

$$(S \xmapsto{q} \mathcal{S}) \longmapsto H^0(Q, \mathcal{O}_Q(n)) = \frac{H^0(\mathbb{P}^2_S, \mathcal{O}(n))}{q \cdot H^0(\mathbb{P}^2_S, \mathcal{O}(n-2))}.$$

Consider the action of $\mathbb{G}_m(S)$ on $H^0(Q, \mathcal{O}_Q(n))$ by multiplication, we get the projectivization $\mathbb{P}(V_n)$ of V_n given by the sheafification of

$$(S \xmapsto{q} \mathcal{S}) \longmapsto (H^0(Q, \mathcal{O}_Q(n)) \setminus \{0\})/\mathbb{G}_m(S).$$

This is equivalent to considering the action of $\mathbb{G}_m$ on $V_n \setminus \sigma_0$ defined by

$$\lambda \cdot (q, f) = (q, \lambda f)$$

where σ_0 is the zero section of $\mathcal{O}_Q(n)$. Accordingly we can also consider the action of $\mathbb{G}_m$ on $\mathbb{A}(1, 2n)\setminus\{0\}$ by multiplication and obtain the projective space $\mathbb{P}(\mathbb{A}(1, 2n))$. Since the torus action and the action of GL_3 commute and $\{0\}$ in $\mathbb{A}(1, 2n)$ is exactly a GL_3-counterpart of the zero section σ_0 in V_n, we have that $V_n \setminus \sigma_0$ is a $\mathrm{GL}_3 \times \mathbb{G}_m$-counterpart of $\mathbb{A}(1, 2n) \setminus \{0\}$. And thus $\mathbb{P}(V_n)$ is a GL_3-counterpart of $\mathbb{P}(\mathbb{A}(1, 2n))$ by taking quotient with respect to the $\mathbb{G}_m$-action.

4. Describe the GL_3 analogue of the discriminant locus in $\mathbb{A}(1, 2n)$ with some explanation.

***Answer*:** Let $\Delta \subset \mathbb{A}(1, 2n)$ be the discriminant locus parametrizing singular binary forms of degree $2n$, *i.e.* binary forms with multiple roots. The discriminant locus Δ is a PGL_2-invariant codimension one closed subscheme of $\mathbb{A}(1, 2n)$ given by a homogeneous polynomial of degree $4n - 2$ in terms of the coefficients $(A_0, \cdots, A_{2n})$ of binary forms that $\mathbb{A}(1, 2n)$ parametrizes.

The goal of this question is to find a GL_3-counterpart of Δ. By the result from **Question 3**, we get V_n as a GL_3-counterpart of the affine space $\mathbb{A}(1, 2n)$. The natural guess is the pairs $(q, f) \in V_n$ such that the closed subscheme $V(q, f) \subset \mathbb{P}^2$ given by the homogeneous ideal (q, f) is singular. This is because the zero locus $Q \subset \mathbb{P}^2$ is smooth, $V(q, f)$ being singular boils down to the form associated to the section f being singular. To be precise, we define the subset

$$D = \{(q, f) \in V_n \mid V(q, f) \subset \mathbb{P}^2 \text{ is singular}\} \subset V_n.$$

The set D indeed has a scheme structure. Here is the reason. We consider the closed subscheme D' of $\mathcal{S} \times \mathbb{A}(2, n) \times \mathbb{P}^2$ defined by

$$\begin{aligned} D' &= \{(q, f, p) \in \mathcal{S} \times \mathbb{A}(2, n) \times \mathbb{P}^2 \mid p \text{ is a singular point of } V(q, f) \subset \mathbb{P}^2\} \\ &= \{(q, f, p) \mid q(p) = f(p) = 0, J_{(q,f)}(p) \text{ does not have maximal rank}\} \end{aligned}$$

where $J_{(q,f)}$ is the Jacobian matrix of (q, f). Thus D inherits the scheme structure from D' along the following projection followed by the quotient map:

$$\mathcal{S} \times \mathbb{A}(2, n) \times \mathbb{P}^2 \to \mathcal{S} \times \mathbb{A}(2, n) \to V_n$$

by considering V_n as the cokernel of the map

$$\begin{aligned} \mathcal{S} \times \mathbb{A}(2, n-2) &\longrightarrow \mathcal{S} \times \mathbb{A}(2, n) \\ (q, g) &\longmapsto (q, qg). \end{aligned}$$

Therefore it is easy to see that $D \to \Delta \times_{\mathcal{M}_0} \mathcal{S}$ given by sending (q, f) to

$$\left(\begin{array}{ccc} Q & \overset{i}{\hookrightarrow} & \mathbb{P}^2_S \\ \downarrow \pi & \swarrow & \\ S & & \end{array} , f \in H^0(Q, \mathcal{O}_Q(n)), i^*\mathcal{O}(1) \cong \mathcal{T}_{Q/S} \right)$$

where as usual, Q is the zero locus of the section $q \in \mathcal{S}$, is an isomorphism. Therefore D is a GL_3-counterpart of the discriminant locus $\Delta \subset \mathbb{A}(1, 2n)$.

5. Use this to give a presentation of the stack $\mathcal{H}_g$ as a quotient by $\mathrm{GL}_3 \times \mathbb{G}_m$.

***Answer*:** First let us recall the following equivalent descriptions of $\mathcal{H}_g$, the moduli stack of genus g smooth hyperelliptic curves. First of all, given a family of hyperelliptic curves $\pi : \mathcal{X} \to S$ over a scheme S, it is equivalent to giving a family of rational curves $p : \mathcal{C} \to S$ over the same base scheme S together with a line bundle $\mathcal{L}$ over $\mathcal{C}$ of degree $-g-1$ and a global section $\sigma \in H^0(\mathcal{C}, \mathcal{L}^{-2})$ with zero locus finite and étale of degree $2g+2$ over the base S. Therefore the pair $(\mathcal{L}, \sigma)$ consists of all the information for getting the double cover $f : \mathcal{X} \to \mathcal{C}$ branched along the zero locus of σ.

In Arsie and Vistoli's paper *Stacks of cyclic covers of projective spaces*, they provide a new description of the stack $\mathcal{H}_g$ as a quotient stack of $\mathbb{A}_{\mathrm{sm}}(1, 2g+2)$ by the group $\mathrm{GL}_2 / \mu_{g+1}$ with action $[A] \cdot f(x, y) = f(A^{-1}(x, y))$. If the genus g is even, then $\mathrm{GL}_2 / \mu_{g+1} \cong \mathrm{GL}_2$ with isomorphism given by sending $[A]$ to $(\det A)^{g/2} A$. And the computation of the integral Chow ring of $\mathcal{H}_g$ has been computed by Edidin and Fulghesu. But in the case that the genus g is odd, we have

$$\mathrm{GL}_2 / \mu_{g+1} \overset{\cong}{\to} \mathrm{PGL}_2 \times \mathbb{G}_m, \quad [A] \mapsto \left([A], (\det A)^{\frac{g+1}{2}} \right).$$

So the stack $\mathcal{H}_g$ is equivalent to the quotient $[\mathbb{A}_{\mathrm{sm}}(1, 2g+2)/(\mathrm{PGL}_2 \times \mathbb{G}_m)]$ with action given by

$$([A], \lambda) \cdot f(x, y) = (\det A)^{g+1} \lambda^{-2} f(A^{-1}(x, y)).$$

To see this more precisely, we start with the prestack over **Sch** consisting of objects over a scheme S as

$$\{ \left(\pi : \mathcal{C} \to S, \phi : \mathcal{C} \cong \mathbb{P}^1_S, \mathcal{L}, \sigma \in H^0(\mathcal{C}, \mathcal{L}^{-2}), \psi : \pi_*(\mathcal{L} \otimes \mathcal{O}(g+1)) \cong \mathcal{O}_S \right) \}$$

where $\mathcal{L}$ is a line bundle over $\mathcal{C}$ of degree $-g-1$. So there is a natural action of $\mathrm{PGL}_2(S)$ on ϕ and an action of $\mathbb{G}_m$ on ψ by multiplication. Therefore this prestack is a $\mathrm{PGL}_2 \times \mathbb{G}_m$-torsor over $\mathcal{H}_g$. And by using the isomorphism $\phi : \mathcal{C} \cong \mathbb{P}^1_S$, the prestack can be further identified with the prestack over **Sch** with objects over S as $\{(\mathbb{P}^1_S \to S, \mathcal{O}(-g-1), \sigma \in H^0(\mathbb{P}^1_S, \mathcal{O}(2g+2)))\}$. This makes all the prestacks mentioned above are all isomorphic to $\mathbb{A}_{\mathrm{sm}}(1, 2g+2)$ and the actions of $\mathrm{PGL}_2 \times \mathbb{G}_m$ are compatible as well.

In Di Lorenzo's paper, he computed the integral Chow ring of $\mathcal{H}_g$ for odd genus g by computing the equivariant Chow ring of a $\mathrm{GL}_3 \times \mathbb{G}_m$-counterpart of $\mathbb{A}_{\mathrm{sm}}(1, 2g+2)$ endowed with the action of $\mathrm{PGL}_2 \times \mathbb{G}_m$ given as above. Using the similar method sketched in **Question 3**, one can show that V_{g+1} is also a $\mathrm{GL}_3 \times \mathbb{G}_m$-counterpart of $\mathbb{A}(1, 2g+2)$ and accordingly, D is a $\mathrm{GL}_3 \times \mathbb{G}_m$-counterpart of the discriminant locus $\Delta \subset \mathbb{A}(1, 2g+2)$. But notice that the $\mathbb{G}_m$-action in the previous results is defined by multiplication with weight 1. But the action of $\mathbb{G}_m$ in the description of $\mathcal{H}_g$ is given by the weight -2. So we should have $V_{g+1} \setminus D$ being a $\mathrm{GL}_3 \times \mathbb{G}_m$-counterpart of $\mathbb{A}_{\mathrm{sm}}(1, 2g+2)$, *i.e.*

$$\mathcal{H}_g \cong [(V_{g+1} \setminus D)/(\mathrm{GL}_3 \times \mathbb{G}_m)]$$

where the $\mathrm{GL}_3 \times \mathbb{G}_m$-action on V_{g+1} is given by

$$(A, \lambda) \cdot (q(\underline{x}), f(\underline{x})) = ((\det A) q(A^{-1}\underline{x}), \lambda^{-2} f(A^{-1}(\underline{x}))), \quad \underline{x} = (x_0, x_1, x_2),$$

and the action of $\mathrm{PGL}_2 \times \mathbb{G}_m$ is given above.

2 QUESTIONS FROM CALIN CHINDRIS

Throughout this section, we will work over $k = \mathbb{C}$.

Let us first recall some definitions and notations. Let $Q = (Q_0, Q_1, t, h)$ be a connected quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$ a dimension vector. Define

(i) $\operatorname{rep}(Q, \mathbf{d}) = \prod_{a \in Q_1} k^{\mathbf{d}(ha) \times \mathbf{d}(ta)}$, and

(ii) $\operatorname{GL}(\mathbf{d}) = \prod_{i \in Q_0} \operatorname{GL}(\mathbf{d}(i), k)$.

The action of $\operatorname{GL}(\mathbf{d})$ on $\operatorname{rep}(Q, \mathbf{d})$ is given by simultaneous conjugation, *i.e.* for any element $g = (g(i))_{i \in Q_0} \in \operatorname{GL}(\mathbf{d})$ and representation $V = (V(a))_{a \in Q_1} \in \operatorname{rep}(Q, \mathbf{d})$, $g \cdot V \in \operatorname{rep}(Q, \mathbf{d})$ is given by

$$g \cdot V(a) = g(ha)V(a)g(ta)^{-1}, \quad \text{for any } a \in Q_1.$$

We define the ring of invariants on $\operatorname{rep}(Q, \mathbf{d})$ to be

$$\begin{aligned} I(Q, \mathbf{d}) &:= k[\operatorname{rep}(Q, \mathbf{d})]^{\operatorname{GL}(\mathbf{d})} \\ &= \{f \in k[\operatorname{rep}(Q, \mathbf{d})] \mid g \cdot f = f, \forall g \in \operatorname{GL}(\mathbf{d})\} \end{aligned}$$

where $\operatorname{rep}(Q, \mathbf{d})$ is an affine space and $k[\operatorname{rep}(Q, \mathbf{d})]$ is its coordinate ring. Moreover, the ring of invariants $I(Q, \mathbf{d})$ is finitely generated over k proved by Hilbert when the group is linearly reductive. Recall that a group G is called linearly reductive if every finitely dimensional G-module is the direct sum of irreducible modules.

Let a linearly reductive group G act on an affine scheme X, we get its induced action on the coordinate ring $\mathcal{O}(X)$ which is finitely generated as a k-algebra. And moreover the invariant subalgebra $\mathcal{O}(X)^G$ is also finitely generated. Under this situation, we get an affine quotient $\pi : X \to X//G$ is a good quotient where $X//G := \operatorname{Spec} \mathcal{O}(X)^G$ is actually an affine scheme. Thus for any point $x \in X$, the orbit closure $\overline{G \cdot x}$ contains a unique closed orbit. This is because if we have two distinct closed orbits W_1, W_2 in $\overline{G \cdot x}$, they map to the same point in $X//G$ under the quotient map π, but this violates π being a good quotient. We have the following theorem proved by Hilbert, Mumford and Kempf:

Theorem 2.1. *Let G be a linearly reductive group and X an affine G-variety. For any $x \in X$, there exists a one-parameter subgroup $\lambda \in X_*(G)$ such that $\lim_{t \to 0} \lambda(t) \cdot x \in C$ where C is the unique closed orbit in the closure $\overline{G \cdot x}$.*

A one-parameter subgroup λ of G is a morphism of algebraic groups $\lambda : \mathbb{G}_m = k^* \to G$. We denote by $X_*(G)$ the set of all one-parameter subgroups of G. For any $x \in X$, we can define a G-map:

$$\begin{aligned} \lambda_x : \mathbb{G}_m &\longrightarrow X \\ t &\longmapsto \lambda(t) \cdot x. \end{aligned}$$

If it can be extended to a morphism $\hat{\lambda_x} : \mathbb{A}^1 \to X$, then we define the limit $\lim_{t \to 0} \lambda_x(t) = \hat{\lambda_x}(0)$. In other words, consider the induced k-algebra homomorphism $\lambda_x^* : k[X] \to k[t, t^{-1}]$, we can express the map $\lambda_x(t) = (P_1(t), \cdots, P_n(t))$ where $P_i(t)$'s are Laurent polynomials in t, for all $i \in [n]$. So

$$\lim_{t \to 0} \lambda(t) \cdot x = \begin{cases} (P_1(0), \cdots, P_n(0)) & \text{if } P_i \in k[t] \\ \infty & \text{otherwise} \end{cases}.$$

The existence of the limit $\lim_{t \to 0} \lambda(t) \cdot x$ means that such limit is a limit point of the orbit $G \cdot x$ and thus it must be contained in the orbit closure $\overline{G \cdot x}$. Theorem 2.1 tells us that the converse is also true, meaning that, if S is a closed G-invariant subset of X satisfying $S \cap \overline{G \cdot x} \neq \emptyset$, then there must be a one-parameter subgroup λ with property $\lim_{t \to 0} \lambda(t) \cdot x \in S$.

We prove the following lemma, which will be used in **Question 2** for obtaining a description of the closed orbits of the $\operatorname{GL}(\mathbf{d})$-action on $\operatorname{rep}(Q, \mathbf{d})$.

Lemma 2.2. *Let $V, W \in \operatorname{rep}(Q, \boldsymbol{d})$, then the followings are equivalent:*

(1) there exists a one-parameter subgroup $\lambda \in X_*(\mathrm{GL}(\boldsymbol{d}))$ such that $\lim_{t\to 0} \lambda(t) \cdot V = W$;

(2) there exists a filtration of finite length of subrepresentations of V:

$$F_\cdot(V) : 0 = V_{m+1} \leq V_m \leq \cdots \leq V_1 \leq V_0 = V$$

such that $\mathrm{gr}_{F_\cdot}(V) = \oplus_{i=0}^m V_i/V_{i+1} \cong W$.

Proof. Assume that $\lim_{t\to 0} \lambda(t) \cdot V \in \operatorname{rep}(Q, \mathbf{d})$ for some $\lambda \in X_*(\mathrm{GL}(\mathbf{d}))$. Recall that for any vertex $x \in Q_0$ there is an induced action of $\mathbb{G}_m$ on the vector space $V(x)$ given by $t \cdot v = \lambda(t)(x)v$ for any $v \in V(x) \cong k^{\mathbf{d}(x)}$ and $t \in \mathbb{G}_m$, where $\lambda(t) \in \mathrm{GL}(\mathbf{d})$ and $\lambda(t)(x) \in \mathrm{GL}(\mathbf{d}(x))$. Define the weight space

$$V_l(x) = \{v \in V(X) \mid t \cdot v = t^l v, \forall t \in \mathbb{G}_m\},$$

we then have the decomposition of $V(x)$ into the weight spaces $V(x) = \oplus_{l\in\mathbb{Z}} V_l(x)$. For any $m \in \mathbb{Z}$, consider the collection of subspaces of $V(x)$,

$$V_{\geq m} := \left(\bigoplus_{k\geq m} V_k(x) \right)_{x\in Q_0}.$$

If we can prove that the existence of $\lim_{t\to 0} \lambda(t) \cdot V$ is equivalent to that $V_{\geq m}$ is a subrepresentation of V for all $m \in \mathbb{Z}$, then in this case we have

$$\lim_{t\to 0} \lambda(t) \cdot V \cong \bigoplus_{m\in\mathbb{Z}} V_{\geq m}/V_{\geq m+1}.$$

So it suffices to show that $\lim_{t\to 0} \lambda(t) \cdot V$ exists is equivalent to that $V_{\geq m}$ is a subrepresentation of V for all $m \in \mathbb{Z}$. For any arrow $a \in Q_1$, the linear map $V(a) : V(ta) \to V(ha)$ can be written as

$$V(a) : \bigoplus_{k\in\mathbb{Z}} V_k(ta) \longrightarrow \bigoplus_{l\in\mathbb{Z}} V_l(ha).$$

Due to the direct sum decomposition of $V(ta)$ and $V(ha)$, the linear map $V(a)$ can be viewed as a block matrix with each (k, l)-block of form

$$V(a)_{k,l} : V_k(ta) \longrightarrow V_l(ha).$$

Recall that k^* acts on $V_k(ta)$ by multiplication of t^k and k^* acts on $V_l(ha)$ by multiplication of t^l. Thus k^* acts on $V(a)_{k,l}$ by multiplication of t^{l-k}.

Therefore $\lim_{t\to 0} \lambda(t) \cdot V$ exists if and only if for any $a \in Q_1$, $V(a)_{k,l} = 0$ whenever $l - k < 0$. It means that $V_k(ta) \to V_l(ha)$ is the zero map if $l < k$, and thus the linear map $V(a)$ restricted on $\oplus_{k\geq m} V_k(ta)$ is an upper triangular block matrix. This is equivalent to

$$V(a) \left(\bigoplus_{k\geq m} V_k(ta) \right) \subseteq \left(\bigoplus_{k\geq m} V_k(ha) \right)$$

which is equivalent to that $V_{\geq m}$ is a subrepresentation of V for each $m \in \mathbb{Z}$. Therefore it yields the result

$$\lim_{t\to 0} \lambda(t) \cdot V = \bigoplus_{m\in\mathbb{Z}} V_{\geq m}/V_{\geq m+1}.$$

Note that $V \in \operatorname{rep}(Q, \mathbf{d})$ is finitely dimensional, then the right-hand-side is indeed a finite direct sum. In other words, there are $M, m \in \mathbb{Z}$, and a finite filtration of V,

$$F_\cdot(V) : 0 = \cdots = V_{\geq M+2} = V_{\geq M+1} \leq V_{\geq M} \leq \cdots \leq V_{\geq m} = V_{\geq m-1} = \cdots = V$$

with $\mathrm{gr}_{F_\cdot}(V) = \lim_{t\to 0} \lambda(t) \cdot V$.

Conversely, suppose there is a filtration

$$F_\cdot(V): 0 = V_{m+1} \leq V_m \leq \cdots \leq V_1 \leq V_0 = V$$

of subrepresentations of V. We want to find a one-parameter subgroup $\lambda \in X_*(\mathrm{GL}(\mathbf{d}))$ such that $\lim_{t\to 0} \lambda(t)\cdot V = \mathrm{gr}_{F_\cdot}(V)$.

For any vertex $x \in Q_1$, take a basis of the vector space $V(x)$ compatible with the filtration of subspaces of $V(x)$, namely $0 = V_{m+1}(x) \leq V_m(x) \leq \cdots \leq V_1(x) \leq V_0(x) = V(x)$. Then for any arrow $a \in Q_1$, we can interpret $V(a)$ as an upper triangular block matrix

$$V(a) = \begin{pmatrix} V_m(a) & X_{1,2}(a) & \cdots & X_{1,m+1}(a) \\ & (V_{m-1}/V_m)(a) & \cdots & X_{2,m+1}(a) \\ & & \ddots & \vdots \\ & & & (V_0/V_1)(a) \end{pmatrix}.$$

Now let's define a one-parameter subgroup $\lambda \in X_*(\mathrm{GL}(\mathbf{d}))$ as follows. For any $x \in Q_0$ and $t \in k^*$,

$$\lambda(t)(x) = \begin{pmatrix} t^m I_{\dim(V_m(x))} & & & \\ & \ddots & & \\ & & t^1 I_{\dim(V_1/V_2(x))} & \\ & & & t^0 I_{\dim(V_0/V_1(x))} \end{pmatrix} \in \mathrm{GL}(\mathbf{d}(x)),$$

which is a change of basis matrix of the vector space $V(x)$. With the change of basis matrix acting by conjugation, we compute

$$(\lambda(t)\cdot V)(a) = \begin{pmatrix} V_m(a) & tX_{1,2} & t^2 X_{1,3} & \cdots & \\ & (V_{m-1}/V_m)(a) & & & \\ & & \ddots & & \\ & & & & (V_0/V_1)(a) \end{pmatrix}$$

where the block matrices on the upper-right corner are all approaching to zero as $t \to 0$. Therefore only the diagonal block matrices are left and thus

$$\lim_{t\to 0} \lambda(t)\cdot V = \mathrm{gr}_{F_\cdot}(V) = \bigoplus_{i=0}^{m} V_i/V_{i+1}. \qquad \square$$

The goal for **Question 3** is to prove the King's semi-stability criterion. The key is to apply the Hilbert-Mumford numerical criterion to characterize χ-semi-stable element on some affine variety for some character χ of G. It is much easier to check semi-stability with King's criterion. Let us first introduce notation and terminology which we will use in the proof.

Keep the same notation as above, let G be a linearly reductive algebraic group and X an affine variety. We denote by $SI(X,G) = k[X]^{[G,G]}$, *i.e.*

$$SI(X,G) = \{f \in k[X] \mid g\cdot f = f, \forall g \in [G,G]\}.$$

A (rational) character χ of group G is a morphism $G \to \mathbb{G}_m$ of algebraic groups. For example, we have that any character of $\mathbb{G}_m$ is defined by $t \mapsto t^r$ for some $r \in \mathbb{Z}$ and thus the set of characters $X^*(\mathbb{G}_m) \cong \mathbb{Z}$. Moreover, characters and one-parameter subgroups form a pairing. Since for any $\chi \in X^*(G)$ and $\lambda \in X_*(G)$, their composition $\chi\circ\lambda : \mathbb{G}_m \to \mathbb{G}_m$ is given by sending t to t^r for some integer $r \in \mathbb{Z}$, it yields a pairing $X^*(G) \times X_*(G) \to \mathbb{Z}$ given by $\langle \chi, \lambda\rangle = r$ if $\chi\circ\lambda(t) = t^r$.

For any character $\chi \in X^*(G)$, we define the space of χ-semi-invariants on X to be

$$SI(X,G)_\chi = \{f \in k[X] \mid g\cdot f = \chi(g)f, \forall g \in G\}.$$

These are also referred to the weight spaces of $SI(X,G)$ and it admits a weight spaces decomposition.

In our situation, we take $X = \text{rep}(Q, \mathbf{d})$ and $G = \text{GL}(\mathbf{d})$. We know that the characters of $\text{GL}(n)$ is completely defined by $g \mapsto \det(g)^r$ for some integer $r \in \mathbb{Z}$. We can further generalize to the description of the characters of $G = \text{GL}(\mathbf{d})$. Recall that we define

$$\text{GL}(\mathbf{d}) = \prod_{i \in Q_0} \text{GL}(\mathbf{d}(i)),$$

so a character of $\text{GL}(\mathbf{d})$ must have form $\prod_{i \in Q_0} \chi(\mathbf{d}(i))$ where each $\chi(\mathbf{d}(i))$ is a character of $\text{GL}(\mathbf{d}(i))$. Moreover each $\chi(\mathbf{d}(i))$ is fully determined by its restriction to the subset of diagonal matrices in $\text{GL}(\mathbf{d}(i))$. This is due to the fact that $\chi(\mathbf{d}(i))$ is a group homomorphism and the target group k^* is abelian, so each $\chi(\mathbf{d}(i))$ should be invariant under the conjugation action of $\text{GL}(\mathbf{d}(i))$ on itself. Thus the character $\chi(\mathbf{d}(i))$ is completely determined by its restriction to the maximal torus $\mathbb{G}_m^{\mathbf{d}(i)}$. We know that the character of $\mathbb{G}_m^{\mathbf{d}(i)}$ has form $(t_1, \cdots, t_{\mathbf{d}(i)}) \mapsto t_1^{\alpha_1} \cdots t_{\mathbf{d}(i)}^{\alpha_{\mathbf{d}(i)}}$, but it is also clear that it should be invariant under the permutation of diagonal entries. It means that all the weights $\alpha_1 = \cdots = \alpha_{\mathbf{d}(i)}$ have to be equal. Therefore, for any character $\chi \in X^*(\text{GL}(\mathbf{d}))$, it can be written as $\chi = \prod_{i \in Q_0} \det^{\theta(i)}$ for some weight $\theta \in \mathbb{Z}^{Q_0}$.

Notice that the above process can also be reversed. Namely, given any integer weight $\theta = (\theta(i))_{i \in Q_0} \in \mathbb{Z}^{Q_0}$, we can defined a character χ_θ associated to the weight θ, given by $\chi_\theta = \prod_{i \in Q_0} \det^{\theta(i)}$. We could simply use the weight θ to represent the character χ_θ induced by it. Therefore, by considering $X = \text{rep}(Q, \mathbf{d})$ and $G = \text{GL}(\mathbf{d})$ with the action of $\text{GL}(\mathbf{d})$ defined by conjugation, we can write

$$\begin{aligned} &SI(Q, \mathbf{d}) = k[\text{rep}(Q, \mathbf{d})]^{\text{SL}(\mathbf{d})}, \\ &SI(Q, \mathbf{d})_\theta = \{f \in k[\text{rep}(Q, \mathbf{d})] \mid g \cdot f = \chi_\theta(g) f, \forall g \in \text{GL}(\mathbf{d})\}. \end{aligned}$$

In general, we define the χ-semi-stable locus of X to be

$$X_\chi^{ss} := \{x \in X \mid \exists f \in SI(X, G)_{\chi^n} \text{ for some } n \in \mathbb{Z}_{>0} \text{ such that } f(x) \neq 0\}.$$

Now we are ready to state our main theorem, the Hilbert-Mumford numerical criterion of elements in X_χ^{ss}:

Theorem 2.3. *Let G be a linearly reductive group and X an affine variety. Let $\chi \in X^*(G)$ be a character of G. For $x \in X$, the followings are equivalent:*

(1) x is a χ-semi-stable point in X;
(2) For any one-parameter subgroup $\lambda \in X_*(G)$, if $\lim_{t \to 0} \lambda(t) \cdot x$ exists, then $\langle \chi, \lambda \rangle \leq 0$.

Finally, let us recall how we define the numerical semi-stability of quiver representations. Let $\theta \in \mathbb{Z}^{Q_0}$ be an integral weight of Q, then we say that $V \in \text{rep}(Q, \mathbf{d})$ is θ-semi-stable if $\theta \cdot \mathbf{d} = 0$ and $\theta \cdot \mathbf{d}_W \leq 0$ for any subrepresentation W of V.

Now we can turn to the proof the King's semi-stability criterion partially stated in **Question 3**.

1. Let Q be the quiver with one vertex and one loop. Explicitly describe the ring of invariants $I(Q, n) = \mathbb{C}[\text{rep}(Q, n)]^{\text{GL}(n)}$ for any positive integer n.

   ***Answer*:** Suppose that Q is the quiver with one vertex and one loop, and $\mathbf{d} = n \in \mathbb{N}$ is a dimension vector which associate the space k^n to the single vertex in Q. Then
   (a) $\text{rep}(Q, n) = k^{n \times n}$ is an affine space consisting of all linear maps from $k^{\mathbf{d}(ta)} \cong k^n$ to $k^{\mathbf{d}(ha)} \cong k^n$ which can be represented by $n \times n$ matrices with entries in k, and
   (b) $\text{GL}(\mathbf{d}) = \text{GL}(n)$, with action on $\text{rep}(Q, n)$ given by for any $g \in \text{GL}(n)$ and $A \in \text{rep}(Q, n)$, $g \cdot A = gAg^{-1}$.

Our goal is to describe the ring of invariants $I(Q,n) = k[\mathrm{Mat}_{n\times n}(k)]^{\mathrm{GL}(n)}$ explicitly. First notice that

$$\begin{aligned} I(Q,n) &= \{f \in k[k^{n\times n}] \mid g\cdot f(A) = f(g^{-1}\cdot A) = f(A), \forall A \in k^{n\times n}, \forall g \in \mathrm{GL}(n)\} \\ &= \{f \in k[k^{n\times n}] \mid f(g^{-1}Ag) = f(A), \forall A \in k^{n\times n}, \forall g \in \mathrm{GL}(n)\} \\ &= \{f \in k[k^{n\times n}] \mid f(gAg^{-1}) = f(A), \forall A \in k^{n\times n}, \forall g \in \mathrm{GL}(n)\}. \end{aligned}$$

The idea is that we can reduce to the subset of diagonalizable matrices in $\mathrm{Mat}_{n\times n}(k)$ since it is dense in the affine space $k^{n\times n}$, and further reduce to the set of diagonal matrices in $\mathrm{Mat}_{n\times n}(k)$. Let us expand this idea below. According to the idea of considering diagonal matrices, we can consider the eigenvalues of a given matrix A, and those elementary symmetric polynomials in terms of eigenvalues are obviously $\mathrm{GL}(n)$-invariant elements in the ring $k[k^{n\times n}]$.

Precisely, consider $\det(tI_n - A)$ as an element in $k[\mathrm{rep}(Q,n)][t]$. The roots for $\det(tI_n - A)$ as a polynomial in terms of t are exactly the eigenvalues of A, and we can expand it as following:

$$\det(tI_n - A) = s_0(A)t^n + (-1)s_1(A)t^{n-1} + \cdots + (-1)^n s_n(A)$$

where $s_0(A) = 1, s_1(A) = \mathrm{tr}(A), s_n(A) = \det(A)$, and in general,

$$s_i(A) = \sum_{1\le j_1<\cdots<j_i\le n} t_{j_1}\cdots t_{j_i}$$

is the i-th elementary symmetric polynomial e_i evaluated at the eigenvalues of A.

We first claim that $s_i \in I(Q,n)$ for each $1 \le i \le n$. Before proving this, let us first recall the fundamental theorem of symmetric function: we have

$$k[t_1,\cdots,t_n]^{\mathfrak{S}_n} = k[e_1,\cdots,e_n]$$

and $e_1,\cdots,e_n$ are algebraically independent over k.

There are multiple different bases for this invariant ring, and let's sketch the proof using double induction. We define the weight of a monomial $X_1^{\alpha_1}\cdots X_n^{\alpha_n}$ in $X_1,\cdots,X_n$ to be $\sum n\alpha_n$. The reason why we define the weight in this way is that once we replace X_i by e_i it will return the right degree. The weight of a polynomial in $k[X_1,\cdots,X_n]$ is the maximal weight of monomials that occur. We want to show the followings.

(a) Given any symmetric polynomial $f(\underline{t}) \in k[t_1,\cdots,t_n]$ of degree d, there exists a polynomial $g(X_1,\cdots,X_n)$ of weight at most d such that $f(\underline{t}) = g(e_1,\cdots,e_n)$.

(b) $\{e_1,\cdots,e_n\}$ are indeed algebraically independent over k.

In order to prove (a), we proceed using double induction on n and the degree d. For $n=1$, we have $t_1 = e_1$. Now assume the assertion holds for $n-1$ variables. Use induction on the degree of f, namely d. The base case when $d=0$ is trivial. Assume $d>0$ and the result holds for $\deg f < d$. Let $f(\underline{t}) \in k[t_1,\cdots,t_n]$ be a degree d polynomial, then $f(t_1,\cdots,t_{n-1},0)$ has degree $d-1$. Thus there exists a $g_1(X_1,\cdots,X_{n-1})$ of weight at most $d-1$ such that

$$f(t_1,\cdots,t_{n-1},0) = g_1(e_1',\cdots,e_{n-1}')$$

where $e_1',\cdots,e_{n-1}'$ are elementary symmetric polynomials in $t_1,\cdots,t_{n-1}$ obtained by setting $t_n = 0$ in e_i. Now set

$$f_1(t_1,\cdots,t_n) := f(t_1,\cdots,t_n) - g_1(e_1,\cdots,e_{n-1}).$$

It is symmetric of degree at most d and $f_1(t_1,\cdots,t_{n-1},0) = 0$ by definition. Thus t_n is a factor of $f_1(t_1,\cdots,t_n)$ but since f_1 is symmetric so it contains each t_i as factor for all $1\le i\le n$. Therefore

$$f_1(t_1,\cdots,t_n) = e_n f_2(t_1,\cdots,t_n)$$

where $\deg f_2 \le d-n < d$. So by the induction hypothesis, there is some $g_2 \in k[X_1,\cdots,X_n]$ with weight at most $d-n$ such that $f_2(t_1,\cdots,t_n) = g_2(e_1,\cdots,e_n)$. And thus

$$f(\underline{t}) = g_1(e_1,\cdots,e_{n-1}) + e_n g_2(e_1,\cdots,e_n).$$

In order to prove $\{e_1,\cdots,e_n\}$ are algebraically independent, suppose they are not, there exists a polynomial $f(\underline{X}) \in k[X_1,\cdots,X_n]$ such that $f(e_1,\cdots,e_n) = 0$. Take a nonzero polynomial with the least degree satisfying this condition, denoted by $f(\underline{X})$. Then we can write f as a polynomial in X_n with coefficients in $k[X_1,\cdots,X_{n-1}]$. Namely,

$$f(\underline{X}) = f_0(X_1,\cdots,X_{n-1}) + f_1(X_1,\cdots,X_{n-1})X_n + \cdots + f_d(X_1,\cdots,X_{n-1})X_n^d$$

where f_0 cannot be zero since otherwise it violates the minimality of the degree of f. Together with $f(e_1,\cdots,e_n) = 0$ and by setting $t_n = 0$, we have $f_0(e'_1,\cdots,e'_{n-1}) = 0$ where $\{e'_1,\cdots,e'_{n-1}\}$ is precisely the elementary symmetric polynomials in $t_1,\cdots,t_{n-1}$. Since f_0 is nonzero in $k[X_1,\cdots,X_{n-1}]$, it implies $\{e_1,\cdots,e_{n-1}\}$ is algebraically dependent, which is a contradiction if we proceed by induction on n.

Now we can prove $I(Q,n) = k[s_1,\cdots,s_n]$ and $\{s_1,\cdots,s_n\}$ is algebraically independent over k. First let

$$D_n = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \middle| \, t_1,\cdots,t_n \in k \right\}$$

and $\mathcal{X}_n$ be the set of all diagonalizable matrices in $\mathrm{Mat}_{n\times n}(k)$, *i.e.*

$$\mathcal{X}_n = \{A \in k^{n\times n} \mid gAg^{-1} \text{ is diagonal for some } g \in \mathrm{GL}(n)\}.$$

Then $\mathcal{X}_n$ is open in $k^{n\times n}$ since it is the complement of the hypersurface defined by the resultant of $m(A), m(A)'$ equaling zero, where $m(A)$ is the minimal polynomial of A. Then $\mathcal{X}_n$ is dense in the affine space $k^{n\times n}$ since the latter is irreducible. So it is reduced to considering the restriction of $f \in k[k^{n\times n}]$ on $\mathcal{X}_n \subset k^{n\times n}$. To further pass to D_n, notice that for each $i \in [n]$,

(a) $s_i|_{D_n} = e_i$ since for a diagonal matrix, the eigenvalues are precisely the diagonal entries.
(b) For any $f \in I(Q,n)$, $f|_{\mathcal{X}_n}$ is completely determined by $f|_{D_n}$. And furthermore, since $\mathcal{X}_n \subset k^{n\times n}$ is dense, f is completely determined by $f|_{D_n}$.
(c) For any $f \in I(Q,n)$, $f|_{D_n} \in k[t_1,\cdots,t_n]^{\mathfrak{S}_n} = k[e_1,\cdots,e_n]$. This is due to the fact that for any permutation $\sigma \in \mathfrak{S}_n$, let $g_\sigma \in \mathrm{GL}_n$ be the permutation matrix associated to σ, we have $f(\sigma \cdot A) = f(g_\sigma A g_\sigma^{-1}) = f(A)$.

Putting everything together, we deduce that

(1) $\{s_1,\cdots,s_n\}$ is algebraically independent since $\{e_1,\cdots,e_n\}$ is algebraically independent over k.
(2) For any $f \in I(Q,n)$, we have $f|_{D_n} = g(e_1,\cdots,e_n)$ for some $g \in k[X_1,\cdots,X_n]$. This is equivalent to

$$\begin{aligned} f|_{D_n} &= g(s_1|_{D_n},\cdots,s_n|_{D_n}) \\ &= g(s_1,\cdots,s_n)|_{D_n} \end{aligned}$$

which is equivalent to $f = g(s_1,\cdots,s_n)$.

Therefore $I(Q,n) = k[s_1,\cdots,s_n]$ and $s_1,\cdots,s_n$ are algebraically independent over k.

2. Let Q be an arbitrary quiver (possibly with oriented cycles), $\mathbf{d}$ is a dimension vector, and $V \in \mathrm{rep}(Q,\mathbf{d})$ a $\mathbf{d}$-dimensional representation of Q. Prove that the $\mathrm{GL}(\mathbf{d})$-orbit of V is closed in $\mathrm{rep}(Q,\mathbf{d})$ if and only if V is a semi-simple representation.

***Answer*:** We will use Lemma 2.2 to get the description of closed orbits.

Let's first assume the orbit $\mathrm{GL}(\mathbf{d}) \cdot V$ is closed in $\mathrm{rep}(Q, \mathbf{d})$, and we want to show that V is a semi-simple representation, *i.e.* it can be expressed as a direct sum of simple representations in $\mathrm{rep}(Q, \mathbf{d})$.

Consider a Jordan-Hölder filtration of the representation V,

$$F.(V) : 0 = V_{m+1} \leq V_m \leq \cdots \leq V_1 \leq V_0 = V$$

with the quotients V_i/V_{i+1} being simple representations for all $i \in \{0, 1, \cdots, m\}$. Then according to Lemma 2.2, there is a one-parameter subgroup $\lambda \in X_*(\mathrm{GL}(\mathbf{d}))$ such that

$$\lim_{t \to 0} \lambda(t) \cdot V = \mathrm{gr}_{F.}(V) = \bigoplus_{i=0}^{m} V_i/V_{i+1}.$$

Since $\lim_{t\to 0} \lambda(t) \cdot V$ is in the closure $\overline{\mathrm{GL}(\mathbf{d}) \cdot V}$, it is contained in $\mathrm{GL}(\mathbf{d}) \cdot V$ by the assumption that the orbit $\mathrm{GL}(\mathbf{d}) \cdot V$ is closed. Therefore $V \cong \lim_{t\to 0} \lambda(t) \cdot V = \oplus_{i=0}^{m} V_i/V_{i+1}$, and V_i/V_{i+1}'s are simple for all $i = \{0, 1, \cdots, m\}$, and thus it implies that V is semi-simple.

Next, conversely, assume V is a semi-simple representation in $\mathrm{rep}(Q, \mathbf{d})$. We want to show that the orbit $\mathrm{GL}(\mathbf{d}) \cdot V$ is closed. Let C be the unique closed orbit of $\overline{\mathrm{GL}(\mathbf{d}) \cdot V}$, if we can prove that $C = \mathrm{GL}(\mathbf{d}) \cdot V$ then we are done. By the above Hilbert-Mumford-Kempf Theorem 2.1 stated at the very beginning of this section, there exists a one-parameter subgroup $\lambda \in X_*(\mathrm{GL}(\mathbf{d}))$ such that $\lim_{t\to 0} \lambda(t) \cdot V \in C$. According to Lemma 2.2 again, it is equivalent to the existence of a finite filtration of subrepresentations of V, say

$$F.(V) : 0 = V_{m+1} \leq V_m \leq \cdots \leq V_1 \leq V_0 = V$$

such that $\lim_{t\to 0} \lambda(t) \cdot V = \mathrm{gr}_{F.}(V)$. Since V is semi-simple, we can view it as a semi-simple $k[\mathrm{GL}(\mathbf{d})]$-module. Thus

$$V \cong \mathrm{gr}_{F.}(V)$$

for any filtration $F.(V)$ of submodules of V. This can be shown by induction on the length of filtration. Namely, since V_1 is a proper submodule of V, and since V is semi-simple, V_1 is a direct summand of V. So we can write $V \cong V_1 \oplus V/V_1$. But we can also write $V_1 = \oplus_{i=1}^{m} V_i/V_{i+1}$ by induction hypothesis, thus $V \cong \oplus_{i=1}^{m} V_i/V_{i+1} \oplus V/V_1$. It implies that

$$V \cong \mathrm{gr}_{F.}(V) = \lim_{t\to 0} \lambda(t) \cdot V \in C.$$

Therefore $\mathrm{GL}(\mathbf{d}) \cdot V = C$ because there is only one single orbit in $\overline{\mathrm{GL}(\mathbf{d}) \cdot V}$. It thus implies the orbit $\mathrm{GL}(\mathbf{d}) \cdot V$ is closed.

3. Let Q be an arbitrary quiver with set of vertices Q_0, $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$ a dimension vector, and $\theta \in \mathbb{Z}^{Q_0}$ an integral weight of Q such that $\theta \cdot \mathbf{d} = 0$. Show that if $V \in \mathrm{rep}(Q, \mathbf{d})$ is χ_θ-semi-stable then V is θ-semi-stable where χ_θ is the rational character of $\mathrm{GL}(\mathbf{d})$ induced by θ.

***Answer*:** Before proving V is θ-semi-stable, let us first prove the following claim. Suppose there exists a one-parameter subgroup $\lambda \in X_*(\mathrm{GL}(\mathbf{d}))$ such that $\lim_{t\to 0} \lambda(t) \cdot V$ exists. Within the same assumption stated in the question, *i.e.* $\theta \in \mathbb{Z}^{Q_0}$ is an integral weight of Q with $\theta \cdot \mathbf{d} = 0$, we want to prove that

$$\langle \chi_\theta, \lambda \rangle = \sum_{m \in \mathbb{Z}} \theta \cdot \mathbf{d}_{V \geq m}.$$

The right-hand-side of the above equality is actually a finite sum since we have seen before that $V_{\geq m} = V$ for all $m \ll 0$ and $V_{\geq m} = 0$ for all $m \gg 0$. For any vertex $x \in Q_0$, after choosing a basis for each weight space $V_m(x) = \{v \in V(x) \mid t \cdot v = t^m v, \forall t \in k^*\}$, we can write $\lambda(t)(x) \in \mathrm{GL}(\mathbf{d}(x))$ as a block diagonal matrix:

$$\lambda(t)(x) = \begin{pmatrix} \ddots & & 0 \\ & t^m I_{\dim V_m(x)} & \\ 0 & & \ddots \end{pmatrix}.$$

This is due to the fact that $[\lambda(t)(x)](V_m(x)) \subseteq V_m(x)$. Thus

$$\begin{aligned}
(\chi_\theta \circ \lambda)(t) &= \prod_{x\in Q_0} \det(\lambda(t)(x))^{\theta(x)} \\
&= \prod_{x\in Q_0} \left(\prod_{m\in\mathbb{Z}} \det\left(t^m I_{\dim V_m(x)}\right)\right)^{\theta(x)} \\
&= \prod_{x\in Q_0} t^{\sum_{m\in\mathbb{Z}} m \dim V_m(x)\theta(x)} \\
&= t^{\sum_{x\in Q_0}\sum_{m\in\mathbb{Z}} m \dim V_m(x)\theta(x)} \\
&= t^{\sum_{x\in Q_0}\sum_{m\in\mathbb{Z}} m\left(\dim V_{\geq m}(x) - \dim V_{\geq m+1}(x)\right)\theta(x)} \\
&= t^{\sum_{x\in Q_0}\sum_{m\in\mathbb{Z}} \dim V_{\geq m}(x)\theta(x)} \\
&= t^{\sum_{m\in\mathbb{Z}}\left(\sum_{x\in Q_0} \dim V_{\geq m}(x)\theta(x)\right)} \\
&= t^{\sum_{m\in\mathbb{Z}} \theta\cdot \mathbf{d}_{V_{\geq m}}}
\end{aligned}$$

in which we can switch the order of the double summation since both are finite sums. Therefore

$$\langle \chi_\theta, \lambda\rangle = \sum_{m\in\mathbb{Z}} \theta \cdot \mathbf{d}_{V_{\geq m}}.$$

Back to the proof of showing V is θ-semi-stable. By the numerical definition of θ-semi-stability, we only need to show that $\theta\cdot\mathbf{d}_W \leq 0$ for any subrepresentation W of V. Let $W \leq V$ be any subrepresentation of V, then we have the filtration $0 \leq W \leq V$ of V. By Lemma 2.2, there exists a one-parameter subgroup $\lambda \in X_*(\mathrm{GL}(\mathbf{d}))$ such that $\lim_{t\to 0}\lambda(t)\cdot V \cong V/W \oplus W$. According to the result proved above, we have

$$\langle \chi_\theta, \lambda\rangle = \theta \cdot \mathbf{d}_W + \theta \cdot \mathbf{d} = \theta \cdot \mathbf{d}_W.$$

Since V is χ_θ-semi-stable and the limit $\lim_{t\to 0}\lambda(t)\cdot V$ exists, by the Hilbert-Mumford numerical criterion, *i.e.* Theorem 2.3, we deduce that $\langle \chi_\theta, \lambda\rangle = \theta \cdot \mathbf{d}_W \leq 0$. So we have shown that V is θ-semi-stable if V is χ_θ-semi-stable.

3 QUESTIONS FROM ZHENBO QIN

1. What is the fundamental group of S^1? How do you prove it?

***Answer*:** Throughout this answer, we denote by I the unit interval $[0,1] \subset \mathbb{R}$. Given a topological space X with a point $x_0 \in X$, the fundamental group $\pi_1(X, x_0)$ is a set consisting of homotopy classes of loops in X based at point x_0. With multiplication defined by composing loops, the set $\pi_1(X, x_0)$ has a group structure with identity element 1 as the constant loop at x_0 and inverse element as loop with opposite direction. We define the unit circle S^1 as

$$S^1 = \{(\cos\theta, \sin\theta) \mid \theta \in \mathbb{R}\} = \{e^{i\theta} \mid \theta \in \mathbb{R}\}.$$

The unit circle S^1 is path-connected, so up to isomorphism, the fundamental group $\pi_1(S^1, x_0)$ does not depend on the choice of the base point x_0. We will write $\pi_1(S^1)$ as the fundamental group of the circle S^1.

Fix a base point $1 \in S^1$, finding $\pi_1(S^1)$ is the same as computing $\pi_1(S^1, 1)$. Notice that there is a group homomorphism

$$\begin{aligned} \phi : \mathbb{R} &\longrightarrow S^1 \\ x &\longmapsto e^{2\pi x} \end{aligned}$$

which is also a continuous open map between topological spaces. In particular, this map ϕ is locally a homeomorphism. Namely $\phi : (-1/2, 1/2) \to S^1 \setminus \{(-1\}$ is a homeomorphism. Using this fact, one can prove the following lifting lemma, which allows us to unwrap the loops in S^1 to a unique path in $\mathbb{R}$ containing the information of the number of times the loop winding around the circle. Let us state the lemmas below without proofs:

Lemma 3.1 (Lifting Lemma)**.** *Let σ be any path in S^1 with initial point 1. Then there exists a unique path σ' in $\mathbb{R}$ with initial point 0 such that $\phi \circ \sigma' = \sigma$.*

Under the existence of the unique lifting σ' of σ in the above lemma, we have

Lemma 3.2 (Covering Homotopy Lemma)**.** *If σ, τ are paths in S^1 both with initial point 1 such that $F : \sigma \simeq \tau$ rel $(0,1)$, then there is a unique map $F' : I \times I \to \mathbb{R}$ such that $F' : \sigma' \simeq \tau'$ rel $(0,1)$ with $\phi \circ F' = F$.*

Now let us define

$$\begin{aligned} \chi : \pi_1(S^1, 1) &\longrightarrow \mathbb{Z} \\ [\sigma] &\longmapsto \sigma'(1) \end{aligned}$$

where $\sigma'(1) \in \mathbb{Z}$ since

$$\sigma(1) = 1 = \phi \circ \sigma'(1) = e^{2\pi\sigma'(1)}$$

implies $\sigma'(1) \in \mathbb{Z}$. This map χ is well-defined since for any homotopic loops σ, τ in S^1 their liftings σ', τ' are homotopic with the same end point $\sigma'(1) = \tau'(1)$ as paths in $\mathbb{R}$.

The map χ is also a homomorphism of groups. To see this, take any two homotopy classes $[\sigma], [\tau] \in \pi_1(S^1, 1)$, and let σ', τ' be their liftings respectively. Define another path τ'' in $\mathbb{R}$ with initial point $\sigma'(1)$ and end point $\sigma'(1) + \tau'(1)$, namely

$$\tau''(s) = \sigma'(1) + \tau'(s), \quad s \in I.$$

Since $\sigma'(1) \in \mathbb{Z}$ we have $\phi \circ \tau'' = \phi \circ \tau' = \tau$ and thus $\sigma'\tau''$ is the lifting of $\sigma\tau$. Then

$$\chi([\sigma][\tau]) = \chi([\sigma\tau]) = (\sigma'\tau'')(1) = \sigma'(1) + \tau'(1) = \chi([\sigma]) + \chi([\tau]).$$

Now it remains to show that the map χ is also a bijection. To prove it is onto, for any integer $n \in \mathbb{Z}$, define a path σ in S^1 rotating the circle for n times (counterclockwisely if

$n > 0$ and clockwisely otherwise) with base point 1,

$$\sigma : I \longrightarrow S^1$$
$$s \longmapsto e^{2\pi i n s}$$

with its lifting σ' given by $\sigma'(s) = ns$. So $\chi([\sigma]) = \sigma'(1) = n$. To prove χ is also one-to-one, for any class $[\sigma] \in \pi_1(S^1, 1)$ with $\chi([\sigma]) = 0$, we have $\sigma'(1) = 0 = \sigma'(0)$. It means that σ' is a loop in R based at the origin. Since $\mathbb{R}$ is contractible, the loop σ' is homotopic to the constant loop based at 0, *i.e.* $\sigma' \simeq 0$ rel $(0, 1)$. By composing with ϕ, we get $\sigma \simeq 1$ rel $(0, 1)$, which means $[\sigma] = 1$ is the identity element in the group $\pi_1(S^1, 1)$.

Therefore we proved that the fundamental group $\pi_1(S^1, 1) \cong \pi_1(S^1) = \mathbb{Z}$.

2. What are the homology and cohomology groups of $\mathbb{RP}^n$ and $\mathbb{CP}^n$? How do you calculate them?

***Answer*:** Throughout this question, we denote by R a commutative and unitary ring. We will write $H_q(X)$ instead of $H_q(X; R)$ as the singular homology of X with coefficients in R if R is not specified. Let's first sketch the definition of the singular homology group of any topological space and then express $\mathbb{RP}^n, \mathbb{CP}^n$ as spherical complexes in order to compute their homology groups.

Given a topological space X, let $S_q(X)$ be the free R-module generated by all the singular q-simplexes, *i.e.* continuous maps $\Delta^q \to X$. Thus any element in $S_q(X)$ can be written as a finite sum $\sum \gamma_\sigma \sigma$ where σ is a singular q-simplex with coefficient $\gamma_\sigma \in R$. It is called a q-chain. We also define the boundary operator $\partial_q : S_q(X) \to S_{q-1}(X)$ by linearly extending the map sending each singular q-simplex σ to $\sum_{i=0}^{q} (-1)^i \sigma^{(i)}$ where $\sigma^{(i)} = \sigma \circ F_i^q$ and F_i^q is the i-th face map.

It can be checked that $\partial_q \circ \partial_{q+1} = 0$ and thus we get a singular chain complex of X, namely,

$$\cdots \xrightarrow{\partial} S_q(X) \xrightarrow{\partial_q} S_{q-1}(X) \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \to 0.$$

The q-th singular homology group of X is defined by

$$H_q(X) := \frac{Z_q(X)}{B_q(X)} = \frac{\mathrm{Ker}(\partial_q : S_q(X) \to S_{q-1}(X))}{\mathrm{Im}(\partial_{q+1} : S_{q+1}(X) \to S_q(X))}$$

where we call $Z_q(X)$ the group of q-cycles in X and $B_q(X)$ the group of q-boundaries in X.

In order to compute the singular homology group of the real and complex projective spaces, we first want to express them as spherical complexes. A spherical complex is obtained by successively attaching cells to what has already been built. Suppose that we have a pair of spaces (X, A) where $A \subset X$, and a topological space Y with a map $f : A \to Y$. By attaching X to Y along f, we mean the quotient space $X \sqcup Y / (x \sim f(x), \forall x \in A)$, denoted by $X \cup_f Y$. In particular, take a collared pair (E^n, S^{n-1}), then attaching an n-cell to Y along $f : S^{n-1} \to Y$ means that we obtain the quotient space $E^n \cup_f Y$. Note that the new pair $(E^n \cup_f Y, Y)$ is also a collared pair, and their relative homology groups are isomorphic, *i.e.*

$$H_q(f) : H_q(E^n, S^{n-1}) \longrightarrow H_q(E^n \cup_f Y, Y) \tag{3.1}$$

is an isomorphism for all $q \in \mathbb{Z}$.

Now let us focus on the collared pair (E^n, S^{n-1}) and map $f : S^{n-1} \to Y$ along which we get the space $Z = E^n \cup_f Y$. First of all, the relative homology of the pair (E^n, S^{n-1}) fits in the following long exact sequence:

$$\cdots \to H_q(S^{n-1}) \to H_q(E^n) \to H_q(E^n, S^{n-1}) \xrightarrow{\partial} H_{q-1}(S^{n-1}) \to \cdots.$$

So the connecting homomorphism

$$\partial : H_q(E^n, S^{n-1}) \longrightarrow H^{\#}_{q-1}(S^{n-1})$$

is an isomorphism when $q \geq 2$ since E^n is contractible and thus $H_q(E^n) = 0$ for all $q \geq 1$. We are using the homology for augmented chain complex to get $H^{\#}_q$ in order to disregard the contribution in H_0 which comes from a single point. Together with the commutative diagram

$$\begin{array}{ccc} H_q(E^n, S^{n-1}) & \xrightarrow{\partial} & H^{\#}_{q-1}(S^{n-1}) \\ \downarrow \cong & & \downarrow H^{\#}_{q-1}(f) \\ H_q(Z, Y) & \longrightarrow & H^{\#}_{q-1}(Y) \end{array},$$

we get the following long exact sequence:

$$\begin{array}{ccccccccccc} \cdots \longrightarrow H_q(Y) & \longrightarrow & H_q(Z) & \longrightarrow & H_q(Y, Z) & \longrightarrow & H^{\#}_{q-1}(Y) & \longrightarrow & H^{\#}_{q-1}(Z) & \longrightarrow & \cdots \\ & & & & \| & \nearrow & & & & & \\ & & & & H_q(E^n, S^{n-1}) & H^{\#}_{q-1}(f) & & & & & \\ & & & & \| & & & & & & \\ & & & & H^{\#}_{q-1}(S^{n-1}) & & & & & & \end{array}.$$

Moreover we know the homology of S^n that $H^{\#}_q(S^n) = 0$ except for $q = n$. Therefore we get the following facts:

(a) For $q \neq n$ and $q \neq n-1$, $H^{\#}_q(Y) \to H^{\#}_q(Z)$ is an isomorphism;

(b) We have the following exact sequence

$$\begin{array}{c} 0 \longrightarrow H^{\#}_n(Y) \longrightarrow H^{\#}_n(Z) \longrightarrow H^{\#}_{n-1}(S^{n-1}) \xrightarrow{H^{\#}_{n-1}(f)} H^{\#}_{n-1}(Y) \longrightarrow H^{\#}_{n-1}(Z) \longrightarrow 0 \\ \qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad R \end{array}.$$

Since both $\mathbb{RP}^n$ and $\mathbb{CP}^n$ can be obtained by successively attaching cells, we can use the above results inductively to get their homology groups. Before doing that, let's first see how we view them as spherical complexes.

For real projective space $\mathbb{RP}^n$, we have the quotient map $f : S^{n-1} \to \mathbb{RP}^{n-1}$ given by identifying the antipodal points. In order to construct $\mathbb{RP}^n$, one first can embed E^n into S^n as upper hemisphere for instance. Then

$$\mathbb{RP}^n = S^n/(x \sim -x) = E^n/(x \sim -x, \forall x \in \partial E^n \cong S^{n-1}).$$

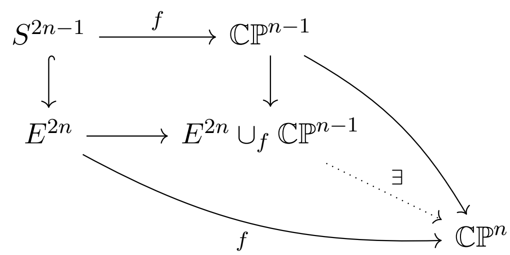
So $\mathbb{RP}^n$ is obtained by attaching an n-cell to $\mathbb{RP}^{n-1}$ along the quotient map $f : S^{n-1} \to \mathbb{RP}^{n-1}$.

For complex projective space $\mathbb{CP}^n$, firstly, we have the map

$$\begin{aligned} f : E^{2n} &\longrightarrow \mathbb{CP}^n \\ (z_0, \cdots, z_{n-1}) &\longmapsto \left(z_0 : \cdots : z_{n-1} : 1 - \left(\sum_{i=0}^{n-1} |z_i|^2 \right)^{1/2} \right). \end{aligned}$$

Then it is easy to see that f sends the boundary ∂E^{2n} to the point $(z_0 : \cdots : z_{n-1} : 0)$, *i.e.* f sends $\partial E^{2n} \cong S^{2n-1}$ into $\mathbb{CP}^{n-1}$. We use the same notation f to denote its restriction to

$\partial E^{2n} \cong S^{2n-1}$. By the following pushout diagram



The map on the bottom-right corner $E^{2n} \cup_f \mathbb{CP}^{n-1} \to \mathbb{CP}^n$, is indeed bijective and continuous. So it turns out to be a homeomorphism since it is defined over a compact space. Follow this construction, $\mathbb{CP}^n$ can be obtained by attaching a $(2n)$-cell to $\mathbb{CP}^{n-1}$ along the map $f : S^{2n-1} \to \mathbb{CP}^{n-1}$.

Now we are ready to compute their homology groups.

Let us start with computing $H_q(\mathbb{CP}^n)$. Claim that

$$H_q(\mathbb{CP}^n) = \begin{cases} 0 & \text{if } q > 2n \text{ or } q \text{ is odd;} \\ R & \text{if } 0 \le q \le 2n \text{ and } q \text{ is even.} \end{cases}$$

In order to prove the claim, use induction of the dimension n. For $n = 0$, we know that $\mathbb{RP}^0$ is a single point and thus $H_0(\mathbb{CP}^0) = R, H_q(\mathbb{CP}^0) = 0$ for all $q \ge 1$. Now if $n \ge 1$, assume that the statement holds for $\mathbb{CP}^{n-1}$. Consider $q \ge 1, n \ge 1$. By the above fact (a), for $q \ne 2n$ and $q \ne 2n-1$, we have

$$H_q(\mathbb{CP}^n) = H_q^{\#}(\mathbb{CP}^n) \cong H_q^{\#}(\mathbb{CP}^{n-1}) = H_q(\mathbb{CP}^{n-1}).$$

Thus we can read off $H_q(\mathbb{CP}^n)$ directly from $H_q(\mathbb{CP}^{n-1})$ when $q \ne 2n, 2n-1$. Now we only need to consider $p = 2n$ and $p = 2n-1$. By fact (b) above, we have the exact sequence

$$\begin{array}{ccccccccccc} 0 \to & H_{2n}^{\#}(\mathbb{CP}^{n-1}) & \to & H_{2n}^{\#}(\mathbb{CP}^n) & \to & H_{2n-1}^{\#}(S^{2n-1}) & \to & H_{2n-1}^{\#}(\mathbb{CP}^{n-1}) & \to & H_{2n-1}^{\#}(\mathbb{CP}^n) & \to 0 \\ & \| & & & & \| & & \| & & & \\ & 0 & & & & R & & 0 & & & \end{array}$$

which implies $H_{2n}^{\#}(\mathbb{CP}^n) = R$ and $H_{2n-1}^{\#}(\mathbb{CP}^n) = 0$. Therefore we have proved the claim.

For computing $H_q(\mathbb{RP}^n)$, claim that

$$H_q(\mathbb{RP}^n) = \begin{cases} 0 & \text{if } q > n; \\ R_2 & \text{if } q \text{ is even and } 1 < q \le n; \\ R/(2) & \text{if } q \text{ is odd and } 1 \le q \le n-1; \\ R & \text{if } q = 0 \text{ or } q = n \text{ and } q \text{ is odd.} \end{cases}$$

We denote by R_2 the set of 2-torsions, *i.e.* $R_2 := \{r \in R \mid 2r = 0\}$.

In particular, if $R = \mathbb{Z}$, we have

$$H_q(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} 0 & \text{if } q > n \text{ or } q \text{ is even and } 1 < q \le n; \\ \mathbb{Z}/2\mathbb{Z} & \text{if } q \text{ is odd and } 1 \le q \le n-1; \\ \mathbb{Z} & \text{if } q = 0 \text{ or } q = n \text{ is odd.} \end{cases}$$

Again, we will proceed using induction on n. If $n = 0$, $H_0(\mathbb{RP}^0) = R$ and $H_q(\mathbb{RP}^0) = 0$ for $q \ge 1$. Consider $q \ge 1$. Now assume $n \ge 1$ and the result holds for $\mathbb{RP}^{n-1}$. If $q \ne n, n-1$, by fact (a), we get $H_q(\mathbb{RP}^n) = H_q(\mathbb{RP}^{n-1})$. If $q = n$ and $q = n-1$, we get an exact sequence by fact (b),

$$0 \to H_n(\mathbb{RP}^n) \to H_{n-1}(S^{n-1}) \cong R \xrightarrow{H_{n-1}(f)} H_{n-1}(\mathbb{RP}^{n-1}) \to H_{n-1}(\mathbb{RP}^n) \to 0. \tag{3.2}$$

So in order to get $H_n(\mathbb{RP}^n)$ and $H_{n-1}(\mathbb{RP}^n)$, we need to know how $H_{n-1}(f)$ is defined. We state the following theorem without proof:

Theorem 3.3. *Let $f : S^n \to \mathbb{RP}^n$ be the canonical quotient map. If n is even, $H_n(f) = 0$. If n is odd, $H_n(f)$ is given by multiplication by 2.*

The theorem can be proved by looking at the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H_n(S^n) & \longrightarrow & H_n(S^n, S^{n-1}) & \longrightarrow & H_{n-1}(S^{n-1}) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & H_n(\mathbb{RP}^n) & \longrightarrow & H_n(\mathbb{RP}^n, \mathbb{RP}^{n-1}) & \longrightarrow & H_{n-1}(\mathbb{RP}^{n-1}) & \longrightarrow \cdots \end{array}$$

where roughly speaking, the middle vertical map can be viewed by pinching S^{n-1} in S^n to a point, then we get the upper half and lower half both homeomorphic to $\mathbb{RP}^n/\mathbb{RP}^{n-1}$ but differed by an antipodal map. Thus it depends on the parity of n.

According the above theorem, back to the exact sequence (3.2), the middle map $H_{n-1}(f)$ is given by multiplication by 2 if n is even, while it is given by the zero map if n is odd. Therefore if n is even, then $H_n(\mathbb{RP}^n) = R_2$ and $H_{n-1}(\mathbb{RP}^n) = R/(2)$. And if n is odd, then $H_n(\mathbb{RP}^n) = R$ and $H_{n-1}(\mathbb{RP}^n) = R_2$. Thus we proved the claim.

The singular cohomology group is defined in the dual manner. We define $S^q(X)$ a R-module consisting of all q-cochains on X with values in R, *i.e.* functions $\mathrm{Sing}_q(X) \to R$. Equivalently, we can view $S^q(X; R) = \mathrm{Hom}(S_q(X), R)$. There is also a coboundary operator $\delta^q : S^q(X; R) \to S^{q+1}(X; R)$ defined by

$$\delta^q(c)(\sigma) = c(\partial_{q+1}\sigma) = \sum_{i=1}^{q+1} (-1)^i c(\sigma \circ F_i^{q+1}),$$

for any q-cochain c and $(q+1)$-simplex σ. One can check that $\delta^q \circ \delta^{q-1} = 0$ is deduced from $\partial_q \circ \partial_{q+1} = 0$. We then define the q-th singular cohomology group of X as the q-th cohomology group of the cochain complex $(S^q(X), \delta^q)$, *i.e.*

$$H^q(X) = \frac{Z^q(X)}{B^q(X)} = \frac{\mathrm{Ker}(\delta^q : S^q(X) \to S^{q+1}(X))}{\mathrm{Im}(\delta^{q-1} : S^{q-1}(X) \to S^q(X))}.$$

Note that on the level of homology and cohomology, there is a Kronecker pair $H^q(X) \times H_q(X) \to R$ which yields a natural homomorphism

$$\alpha : H^q(X) \longrightarrow \mathrm{Hom}(H_q(X), R)$$

defined by $\alpha([c])([\sigma]) = \langle c, \sigma \rangle = c(\sigma)$. The Universal Coefficient Theorem for singular (co)homology tells us under what assumptions the canonical homomorphism α is surjective and what the kernel looks like. Briefly, given a chain complex $C.$ of free abelian groups and an abelian group G, we can write $C^{\cdot} = Hom_{\mathbf{Ab}}(C.; G)$ as the associated cochain complex. Then the theorem tells us that there is a split short exact sequence:

$$0 \to \mathrm{Ext}(H_{q-1}(C.), G) \to H^q(C^{\cdot}; G) \xrightarrow{\alpha} \mathrm{Hom}(H_q(C.), G) \to 0.$$

The result of the theorem can be generalized from abelian groups ($\mathbb{Z}$-modules) to the category of R-modules as long as R is a PID. This is because if R is a PID, then every submodule of a free R-module is free. And moreover, we can take trivial resolution of any free R-module to kill all the Ext groups, *i.e.* length two resolutions always exist, so all high Ext groups vanish.

In particular, consider the integral coefficients, *i.e.* $R = \mathbb{Z}$. Let $T_q \subset H_q(X)$ be the torsion part, then

$$H^q(X;\mathbb{Z}) \cong (H_q(X)/T_q) \oplus T_{q-1}.$$

Now we can write down the singular cohomology group of $\mathbb{CP}^n$ and $\mathbb{RP}^n$ from their homology groups:

$$H^q(\mathbb{CP}^n;\mathbb{Z}) = \begin{cases} 0 & \text{if } q > 2n \text{ or } q \text{ is odd;} \\ \mathbb{Z} & \text{if } 0 \leq q \leq 2n \text{ and } q \text{ is even,} \end{cases}$$

and

$$H^q(\mathbb{RP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } q \text{ is even and } 2 \leq q \leq n; \\ \mathbb{Z} & \text{if } q = 0 \text{ or } q = n \text{ and } q \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases}$$

3. What are the Euler characteristics of $\mathbb{RP}^n$ and $\mathbb{CP}^n$?

***Answer*:** Let X be a topological space. The q-th Betti number, denoted by β_q, is the rank of the abelian group $H_q(X;\mathbb{Z})$. The Euler characteristic of X is defined by

$$\chi(X) = \sum_{q \geq 0} (-1)^q \beta_q$$

whenever the sum is finite.

Recall the results from **Question 2** that

$$H_q(\mathbb{CP}^n;\mathbb{Z}) = \begin{cases} 0 & \text{if } q > 2n \text{ or } q \text{ is odd;} \\ \mathbb{Z} & \text{if } 0 \leq q \leq 2n \text{ and } q \text{ is even.} \end{cases}$$

Then $\beta_q = 0$ when q is odd or $q > 2n$ and $\beta_q = 1$ when $0 \leq q \leq 2n$ and q is even. So the Euler characteristic of $\mathbb{CP}^n$ is $\chi(\mathbb{CP}^n) = n + 1$.

Since

$$H_q(\mathbb{RP}^n;\mathbb{Z}) = \begin{cases} 0 & \text{if } q > n \text{ or } q \text{ is even and } 1 < q \leq n; \\ \mathbb{Z}/2\mathbb{Z} & \text{if } q \text{ is odd and } 1 \leq q \leq n-1; \\ \mathbb{Z} & \text{if } q = 0 \text{ or } q = n \text{ is odd,} \end{cases}$$

it implies $\beta_q = 1$ if $q = 0$ or $q = n$ is odd, and $\beta_q = 0$ otherwise. Therefore the Euler characteristic of $\mathbb{RP}^n$ is

$$\chi(\mathbb{RP}^n) = \begin{cases} 1 & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$