A TEST FAMILY OF HYPERELLIPTIC CURVES

ZHENGNING HU

This short note provides a construction of one-parameter family of hyperelliptic curves with at worst a single node. The test curve constructed here was previously used for a sanity check of the results in [1].

1. Background

Let $\mathcal{H}_g$ be the moduli stack of smooth hyperelliptic curves of genus g, which is a closed smooth substack of $\mathcal{M}_g$, and we denote by $\overline{\mathcal{H}}_g$ its closure in $\overline{\mathcal{M}}_g$. Likewise, we consider $\overline{\mathcal{H}}_{g,n}$ to be the stack of n-pointed stable hyperelliptic curves, defined as the stack-theoretic inverse image of $\overline{\mathcal{H}}_g$ under the forgetful morphism $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$. Equivalently, $\overline{\mathcal{H}}_{g,n}$ is the fiber product $\overline{\mathcal{H}}_g \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g,n}$.

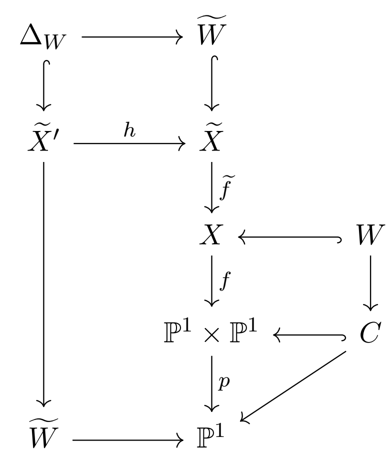
The universal hyperelliptic curve $\mathcal{H}_{g,1}$ contains a substack $\mathcal{H}_{g,w}$ which parametrizes Weierstrass points in the fibers of the morphism $\mathcal{H}_{g,1} \to \mathcal{H}_g$. The substack $\mathcal{H}_{g,w}$ is étale over $\mathcal{H}_g$ of degree $2g+2$ and thus $\mathcal{H}_{g,w}$ is a smooth Cartier divisor in $\mathcal{H}_{g,1}$. We further denote by $\overline{\mathcal{H}}_{g,w}$ its closure in $\overline{\mathcal{H}}_{g,1}$. Note that $\overline{\mathcal{H}}_{g,w}$ is also a (irreducible) Cartier divisor in $\overline{\mathcal{H}}_{g,1}$. Since every stable family of hyperelliptic curves admits a global hyperelliptic involution τ, the restriction of the divisor $\overline{\mathcal{H}}_{g,w}$ to a 1-pointed stable family $X \to B \to \overline{\mathcal{H}}_{g,1}$ is the fixed locus of τ minus the separating nodes of type Δ_i for $i > 0$. The observation can be applied to compute the degree of $\overline{\mathcal{H}}_{g,w}$ on a 1-pointed family $(X \to B, \sigma)$ parametrized by a smooth projective variety B which maps to $\overline{\mathcal{H}}_{g,1}$.

Proposition 1.1. *Given a pointed family of stable genus g hyperelliptic curves $(X \to B, \sigma)$ with the hyperelliptic involution τ, such that $X \to B$ remains stable after forgetting the section σ. The pullback of $\overline{\mathcal{H}}_{g,w}$ along the family $B \to \overline{\mathcal{H}}_{g,1}$ is given by the divisor $\sigma^*\tau^*\mathcal{O}_X(\sigma)$.*

2. Explicit construction

In this section, we give an explicit construction of a one-parameter family $(\widetilde{X}' \to \widetilde{W}, \Delta_W)$ of genus g stable hyperelliptic curves with at worst a nonseparating node, together with a Weierstrass section Δ_W.

Let's start with the projection $p : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ to the second component.



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Take C to be a general smooth curve contained in the linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2g+2,2)|$ such that the restriction $p\,|_C\colon C\to\mathbb{P}^1$ is a simple cover of constant degree $2g+2$. It can be easily checked that the genus of C is $g(C)=2g+1$ and $p\,|_C$ has $8g+4$ ramification points by Riemann-Hurwitz. In order to construct a family of hyperelliptic curves, we take $f:X\to\mathbb{P}^1\times\mathbb{P}^1$ to be the double cover branched over the curve C, and let $W=f^{-1}(C)$ be such that $f^*C=2W$. We are allowed to take double cover of $\mathbb{P}^1\times\mathbb{P}^1$ with branch locus given by C since C is a square in $\mathrm{Pic}(\mathbb{P}^1\times\mathbb{P}^1)$. Thus by projection formula it is easy to see that $f_*W=C$ since $2f_*W=f_*f^*C=2C$. Next we take $\widetilde{f}:\widetilde{X}\to X$ to be the blowup of X at $8g+4$ smooth points on W which map to the ramification points on C, and let $\widetilde{W}$ denote the strict transform of W. We then obtain an unramified cover $\widetilde{W}\to\mathbb{P}^1$ of degree $2g+2$, along which we take a base change $h:\widetilde{X}'\to\widetilde{X}$. Let $\Delta_W=h^{-1}(\widetilde{W})$, and as before, we have $h^*\widetilde{W}=(2g+2)\Delta_W$ and $h_*\Delta_W=\widetilde{W}$. Stabilizing the fibers if necessary, therefore we obtain a family of stable hyperelliptic curves $\widetilde{X}'\to\widetilde{W}$ with a section Δ_W, which only intersects nontrivially with a single boundary divisor η_{irr} described in [1, Proposition 3.5].

3. Computations of intersection numbers

By Proposition 1.1, we see that

$$\deg_{\widetilde{W}}[\overline{\mathcal{H}}_{g,w}]=\Delta_W\cdot\left(\Delta_W+\sum_{i=1}^{8g+4}E_i\right)=(\Delta_W\cdot\Delta_W)+(8g+4),$$

where E_i's are the exceptional $\mathbb{P}^1$'s from the blowup $\widetilde{f}$ of smooth points. And moreover, we also have

$$\deg_{\widetilde{W}}\psi=-(\Delta_W\cdot\Delta_W),\quad \deg_{\widetilde{W}}\eta_{irr}=(2g+2)(8g+4).$$

Therefore it suffices to find the intersection $\Delta_W\cdot K_{\widetilde{X}'}$ as $\Delta_W\cdot\Delta_W=4g-\Delta_W\cdot K_{\widetilde{X}'}$ by the adjunction formula. It can be computed that

$$\begin{aligned}\Delta_W\cdot K_{\widetilde{X}'}&=h_*(\Delta_W\cdot h^*K_{\widetilde{X}})=\widetilde{W}\cdot K_{\widetilde{X}}=\widetilde{W}\cdot\left(\widetilde{f}^*K_X+\sum_{i=1}^{8g+4}E_i\right)\\&=W\cdot K_X+(8g+4)=C\cdot K_{\mathbb{P}^1\times\mathbb{P}^1}+W\cdot W+(8g+4)=8g,\end{aligned}$$

where the last equality is due to the fact that

$$W\cdot W=\frac{1}{4}(f^*C\cdot f^*C)=\frac{1}{4}(C\cdot f_*2W)=\frac{1}{2}C\cdot C=4g+4.$$

Therefore, we obtain the following degrees restricted on this family $(\widetilde{X}'\to\widetilde{W},\Delta_W)$:

$$\deg_{\widetilde{W}}[\overline{\mathcal{H}}_{g,w}]=4g+4,\quad \deg_{\widetilde{W}}\psi=4g,\quad \deg_{\widetilde{W}}\eta_{irr}=8(g+1)(2g+1),$$

which together satisfy the expression of $[\overline{\mathcal{H}}_{g,w}]$ in the divisor class group $\mathrm{Cl}(\overline{\mathcal{H}}_{g,1})$ [1, Theorem 1.1]:

$$\begin{aligned}[\overline{\mathcal{H}}_{g,w}]\ =\ &\left(\frac{g+1}{g-1}\right)\psi-\frac{1}{2(2g+1)(g-1)}\eta_{irr}+\\&\sum_{i=1}^{\lfloor(g-1)/2\rfloor}\left[-\frac{(i+1)(2i+1)}{(2g+1)(g-1)}\eta_{i,0}-\frac{(g-i)[2(g-i)-1]}{(2g+1)(g-1)}\eta_{i,1}\right]+\\&\sum_{i=1}^{\lfloor g/2\rfloor}\left[-\frac{2i(2i+1)}{(2g+1)(g-1)}\delta_{i,0}-\frac{2(g-i)[2(g-i)+1]}{(2g+1)(g-1)}\delta_{i,1}\right].\end{aligned}$$

References

[1] D. Edidin and Z. Hu. Chow classes of divisors on stacks of pointed hyperelliptic curves. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 25(1):217–240, 2024. doi:10.2422/2036-2145.202204_001.